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Modeling and Simulations

EE 562

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Course contents

- System modeling
- Definitions
  - Engineering control problem
  - Types of models
  - System model classifications
  - Modeling steps.
- Review of Linear system and non-linear systems.
- Modeling the physical systems using physical relationships.
- Linearized of non-linear systems
- System representation
- State variable representations.
- Types of state-space representation.
- Controllability
- Observability.
- Linearized of non-linear systems using state space methods
- System simulation (Matlab-simulation)
- System identification.
- Model the physical systems using system identification

The Engineering Control Problem

Control problem can be stated as follows:

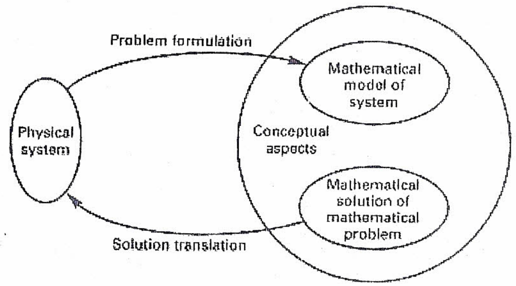
A physical system or process is to be accurately controlled in order that certain control criteria, or specifications, be satisfied.

In general, solving a control problem involves the following steps.

- A set of performance specifications is established (requirements).
- As a results of the performance specifications a control problem may be exists.

To solve the problem and additional steps are required ;

- **A set of differential equations (Modeling) that describe the physical system is formulated.**
- **Using the classical control-theory approach to analysis the system results (Simulation).**
- Accordingly, Control system Design must be implemented to improve the system
- Using the modern control –theory approach , the designer specifies an optimal performance index for the system (i.e the designer yields the necessary structure to minimize the specified performance index, thus producing an optimal system.
- If the physical tests are unsatisfactory, iterating the above steps.
- The process can be summarized as in the following schematic diagram



System modelling

- **Modelling:** is the description of the behavior of physical system by mathematical approach.
- These mathematical equations must be
  - Not too simplified
  - Not too complicated.
- The results of these models must be reasonably accurate.
- Types of Models
- A system model can be classified into
  - Static Model :

(1) the system whose some of its elements behavior fast relative to other elements is often modeled as static model. For example , the switch time of a thermostat is fast compared to the time required for the room temperature to change appreciably. Thus, the thermostat is modeled as a static model.

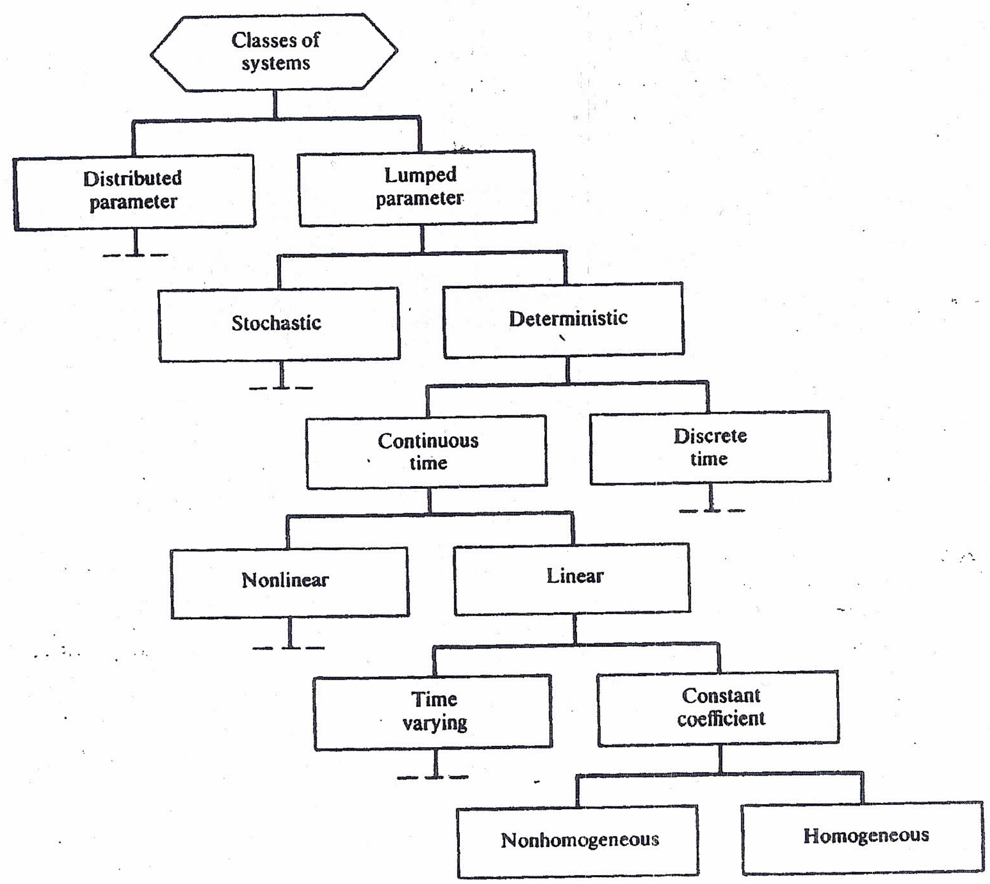
(2) the system whose output at any time depends on only the input at that time. For example, the resistance may be considered as static element, because its current depends on only the voltage applied at present and not on past voltage.

- Dynamic Model :

it is the model whose present output depends on past inputs. For example , the present position of a car depends on where it started and what its velocity has been from the start.

System model classifications

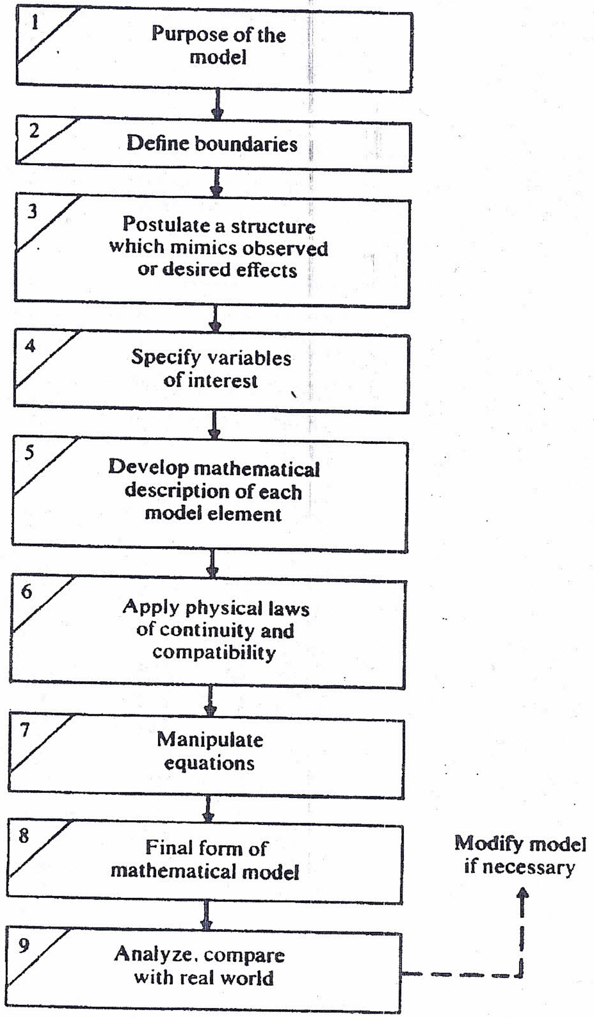
- System models are classified according to the types of equations used to described them. The following family tree illustrates the major system classifications.



Major classes of system equations.

Modeling steps

- An outline of the analytical approach to model physical systems can be represented by the following steps



(b) Steps in modeling

Linear systems (summary)

1. In practical most physical systems are non-linear. They are often approximately modeled by linear equations mainly for mathematical simplicity. This simplification is satisfactory as long as the resulting solutions are in agreement with experimental results.
2. For linear systems the concept of a transfer function can be used.
3. A linear systems in state space can be represented by the form $\dot{x} = Ax + Bu$
4. Linear systems are a systems to which the principle of superposition applies.
5. A linear systems can be analyzed in time or frequency domain.
6. For linear systems a sinusoidal input leads to a sinusoidal output with the same frequency.
7. There is only one equilibrium point in linear system $\dot{x} = 0$)
8. The stability concept depends on the system poles locations .

Non-linear systems

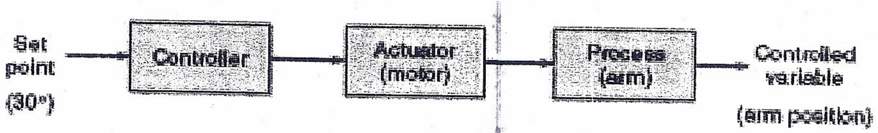
- The non-linear systems differ from linear systems greatly in that the principle of superposition can not be applied.
- Non-linear systems have many phenomena that can not be seen in linear systems. such as :

1. the dependence of the non-linear systems behavior upon the magnitude and type of the input. (i.E the nonlinear system behave completely differently in response to step inputs of different magnitudes.
2. Stability of the system is no longer just a simple function of system poles location but depends on the number and position of system **Equilibrium points**.
3. The correlation between transfer function pole and zero locations and time response behavior is generally invalid.
4. Finite escape time: The state of an unstable linear system goes to infinity as time approaches infinity ; a nonlinear system's state , however, can go to infinity in finite time

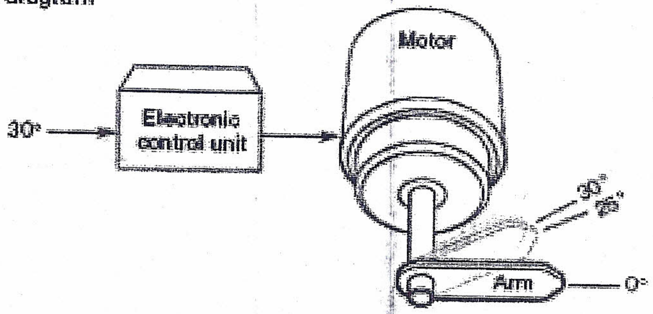
For many nonlinear systems which have sufficiently small and smooth nonlinearities can be treated by using equivalent linearization techniques such as a linear approximate model, obtained by linearizing about a known nominal operating point.

Open and close loop methods

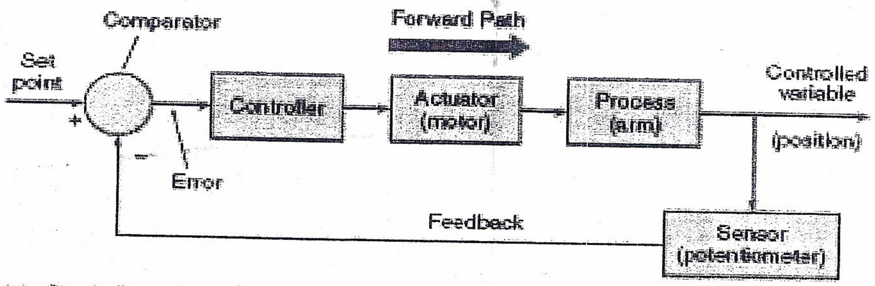
- Open loop



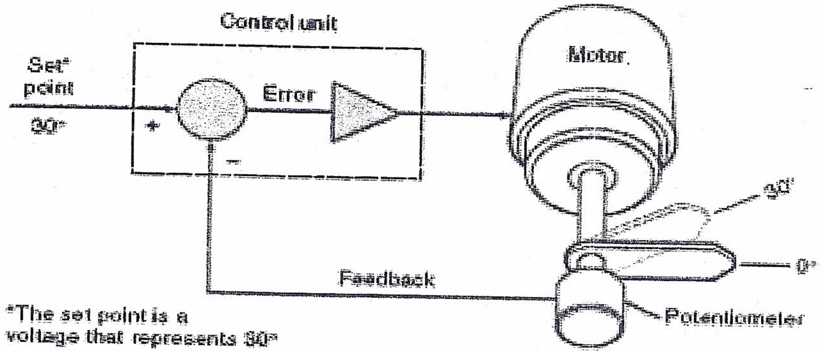
(a) Block diagram



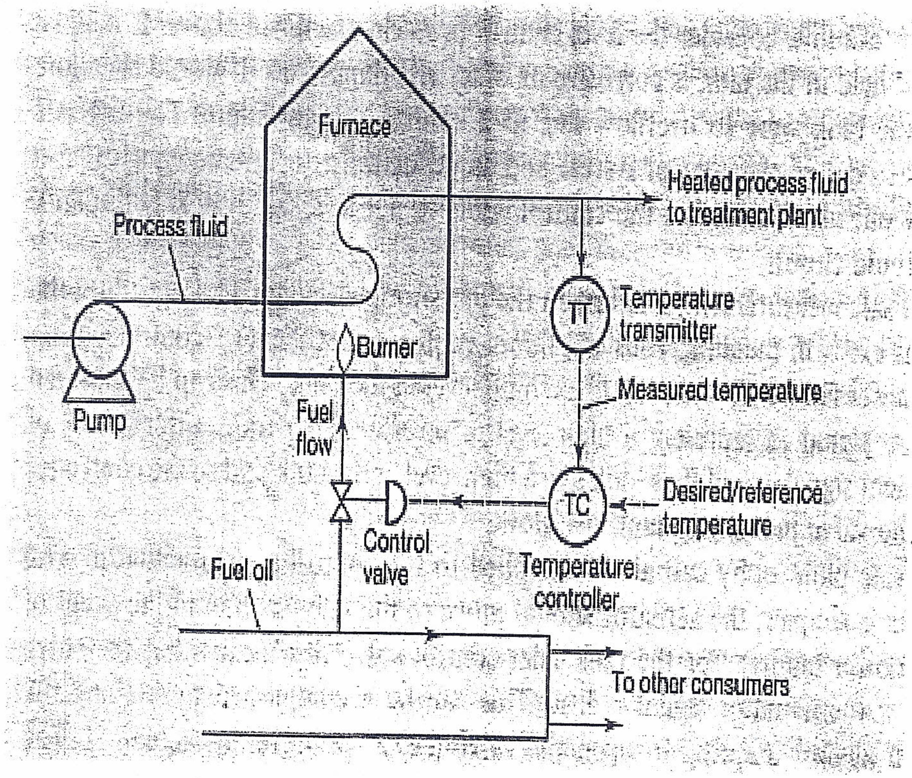
- Close loop

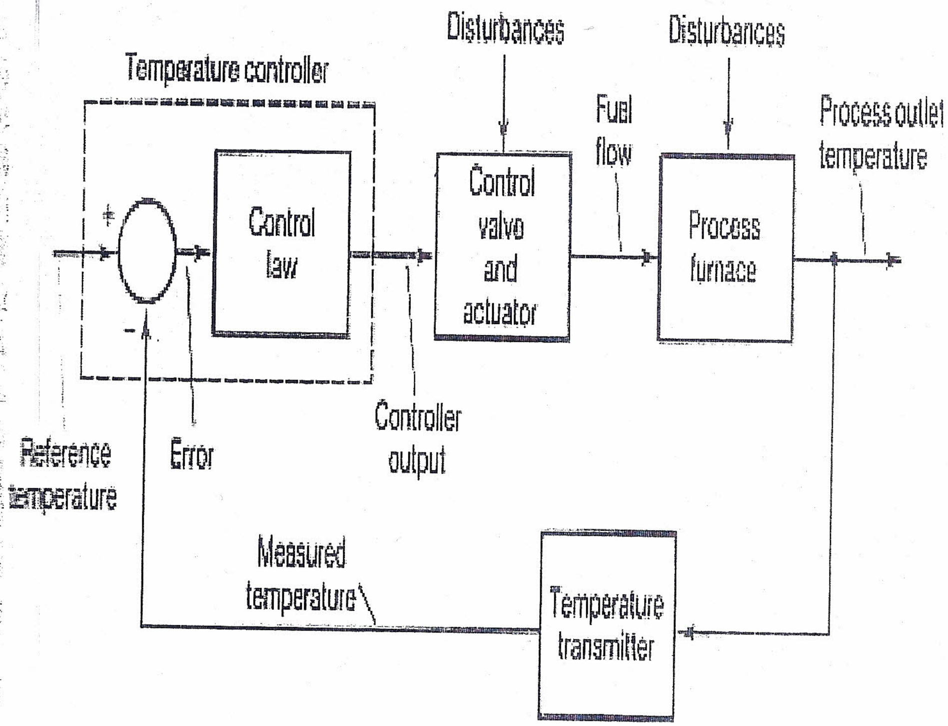


(a) Block diagram

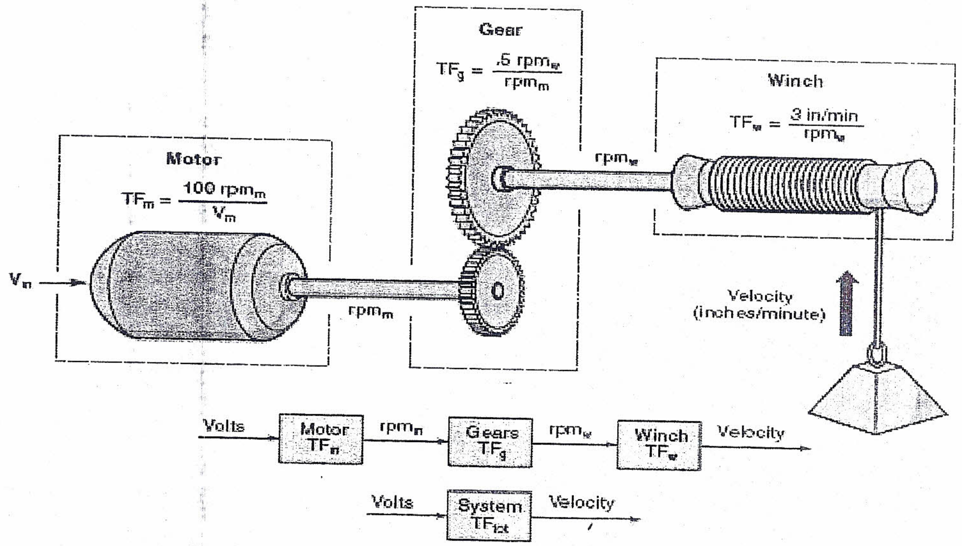


- Temperature control

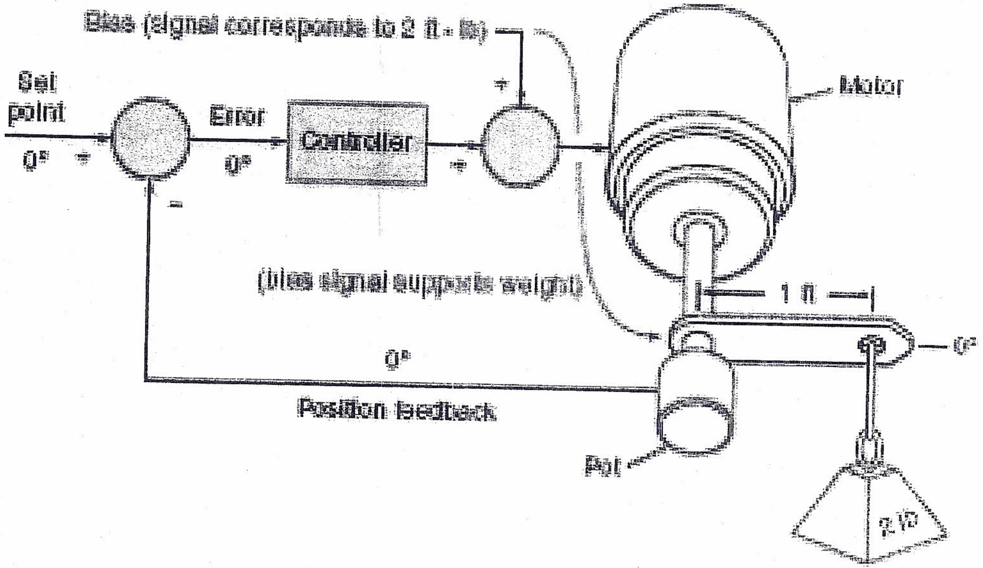




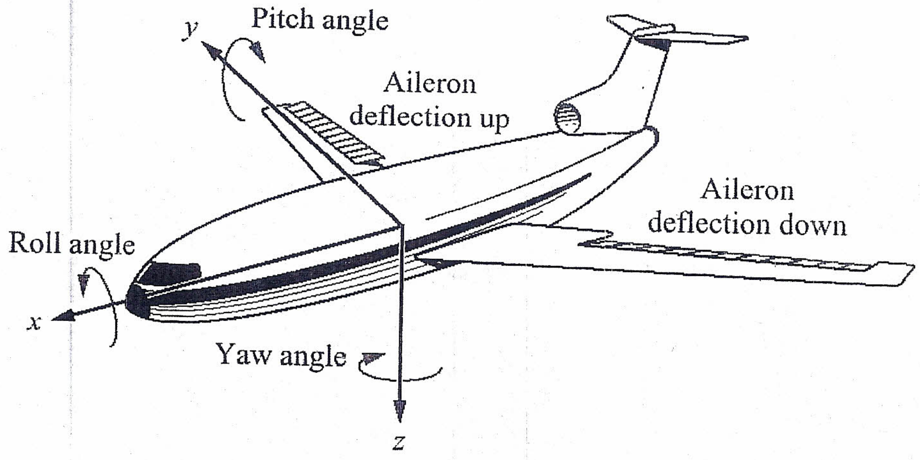
- Wight control (open loop)

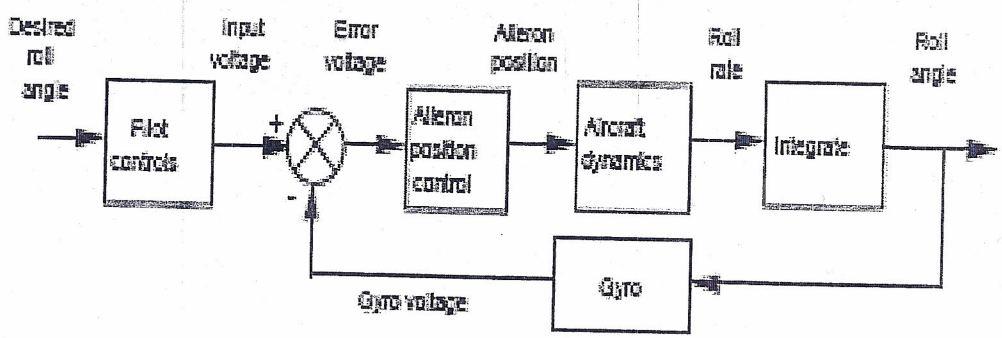


- Wight control (close loop)

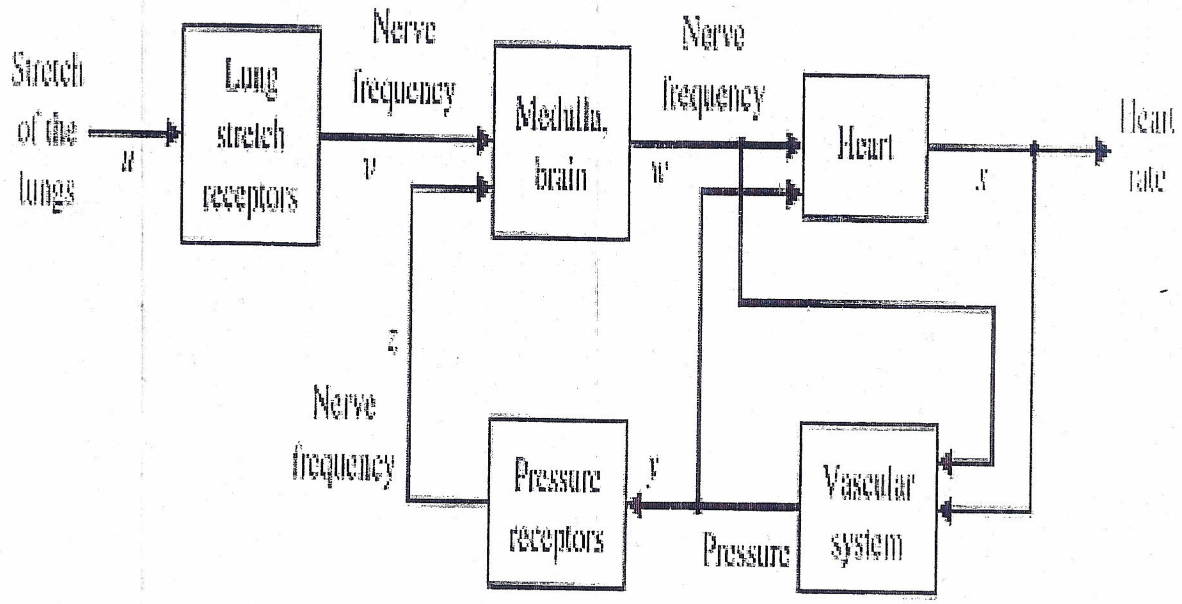


- An aircraft's attitude control (roll angle stabilize)

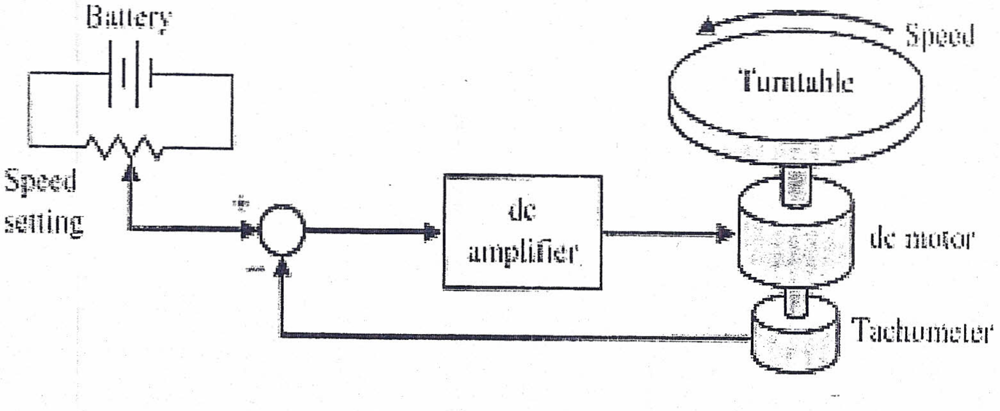




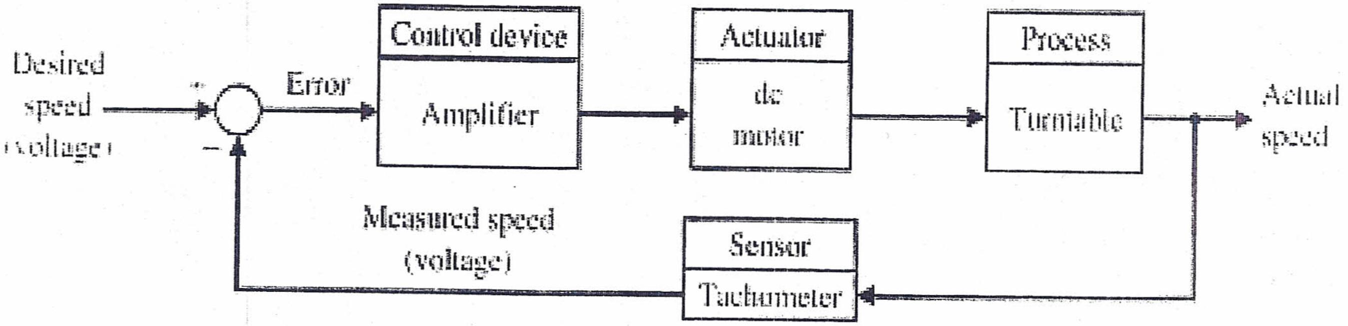
Hart-rate control



- Control of the speed of a turntable

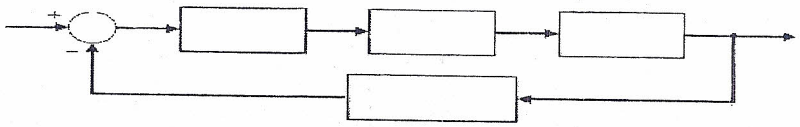


(a)

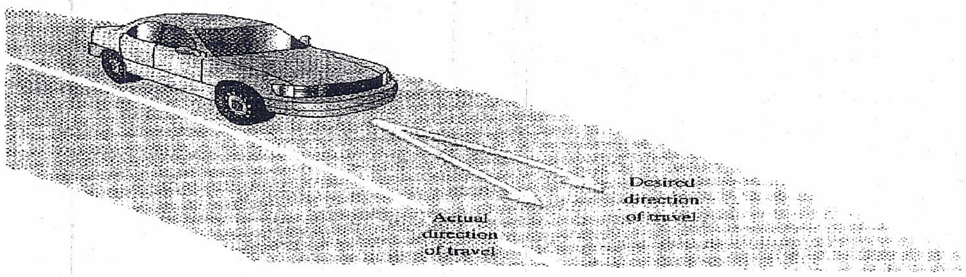


Problems

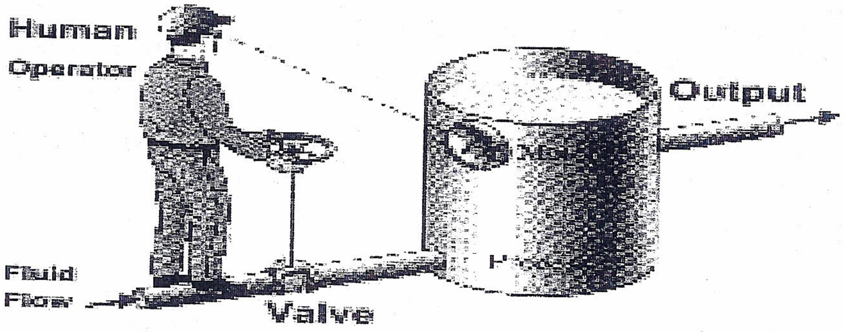
1. An automobile driver uses a control system to control steering of the car on the desired direction Sketch the block diagram for this system.



(a)



2. A control system shown used a human operator as part of a closed loop system . Sketch the block diagram for this system.

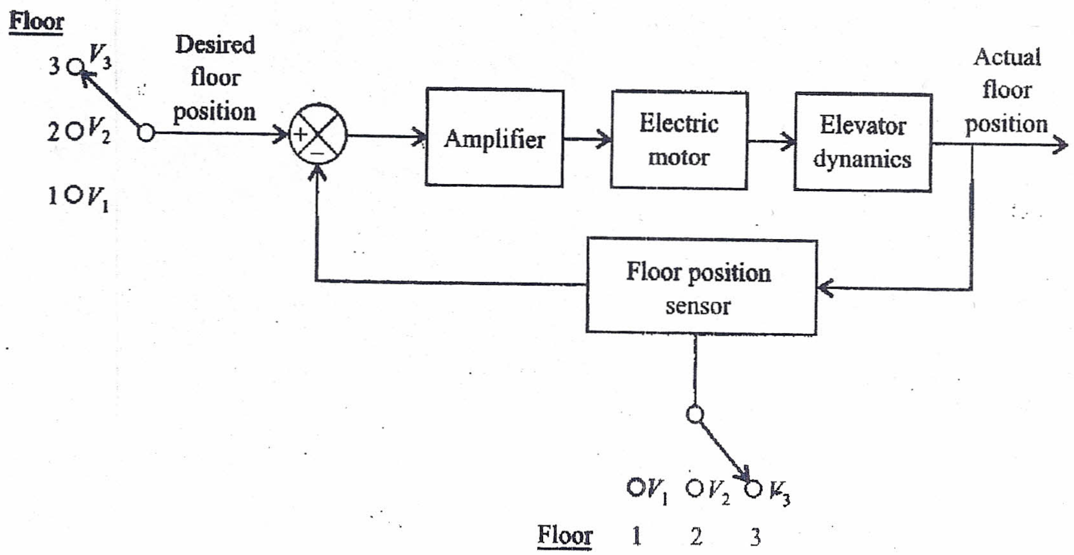


3. The student –teacher learning process is inherently a feedback process intended to reduce the system error to a minimum. Construct a feedback model .

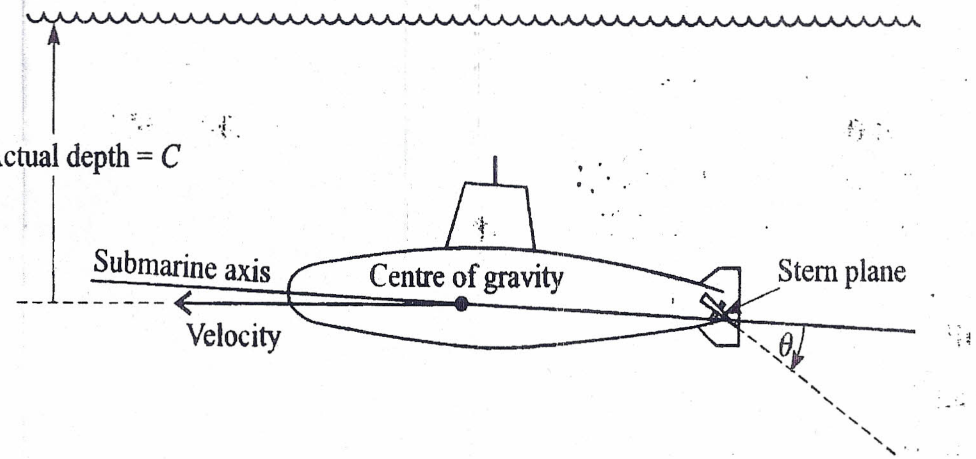
4. Draw a block diagram representation of a thermostatically controlled electric oven in the kitchen of a home.

5. An elevator – position control system is used in an apartment building . Draw the block diagram representation of the elevator in a three-floor building which obtains the desired floor reference position as a voltage from the elevator passenger pressing a button on the elevator, and compares this voltage with a voltage from a position sensor that represents the actual floor position the elevator is at. The difference in an error voltage which is amplified and connected to an electric motor that positions the elevator car to the desired floor selected.

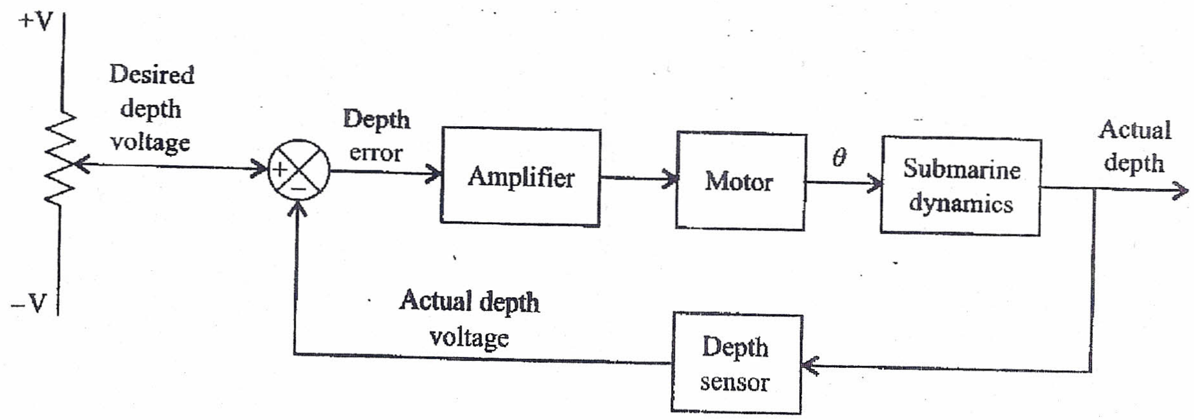
Solution



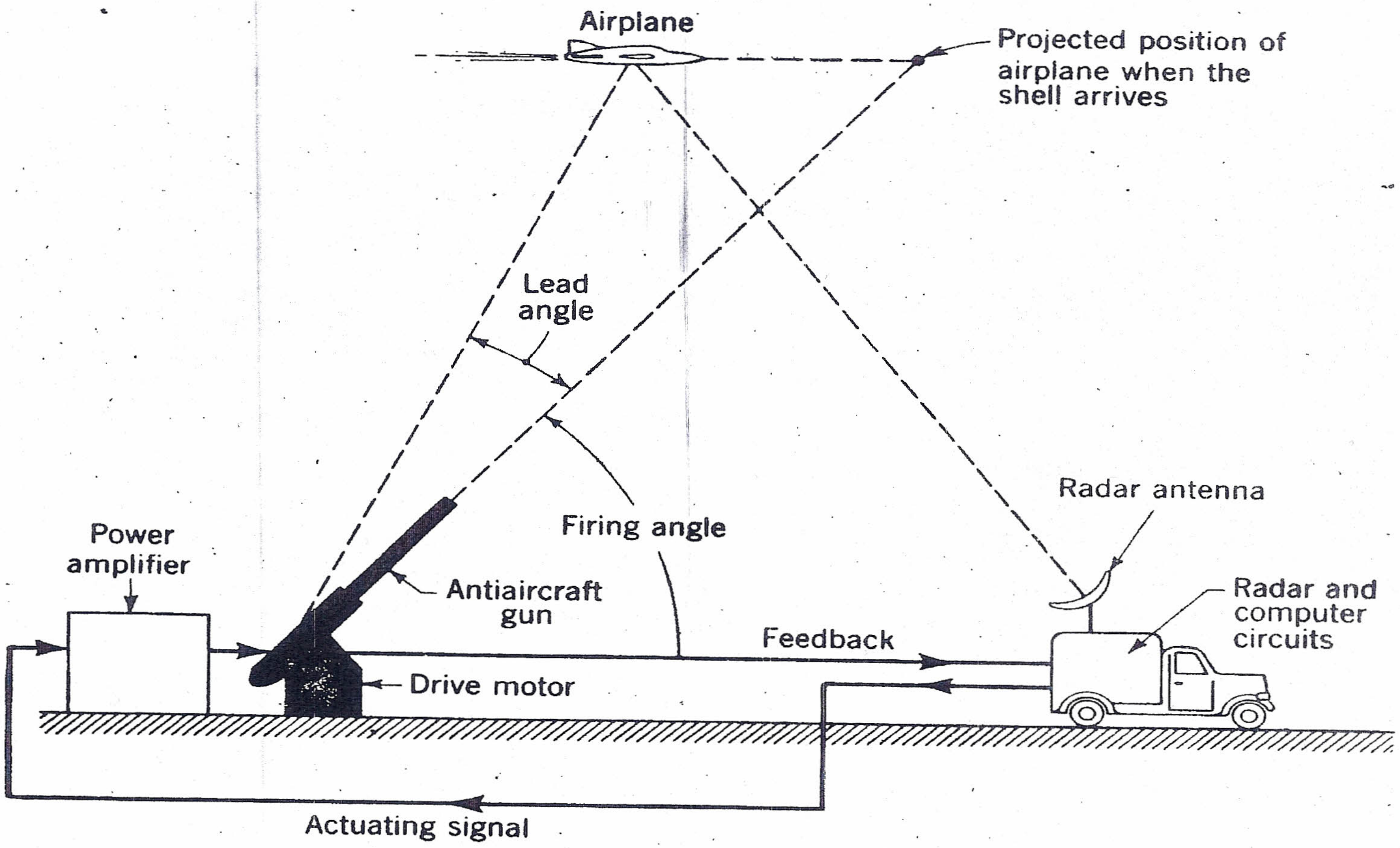
6. The automatic depth control of a submarine given below is an interesting control problem. The desired depth is set as voltage from a calibrated potentiometer. The actual depth is measured by a pressure transducer. Draw the block diagram representation for this system



Solution



6. For antiaircraft radar tracking control system shown below . Sketch the block diagram for this system

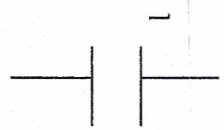


Mathematical models of physical systems

- Most of physical systems are characterized by the term mathematical model
  - Mathematical model is defined as a set of equations that represents the dynamics of physical systems.
- To find mathematical model the following informations must be provided
  1. Appropriate basic physical laws.
  2. The system elements.
  3. The way the system elements are interconnected
- Linearizing the non-linear systems (for certain case only).

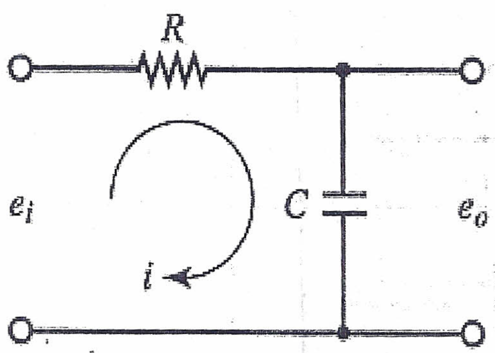
Electrical systems

- The basic system elements are

Elements	Elements model	Model equations
• Resistance (R)		$V = I * R$
• Inductance (L)		$V_L = L \frac{di}{dt}$
• Capacitance (C)		$V_C = \frac{1}{C} \int_0^t I(t) dt$
• The basic physical laws • Ohm's law. • Kirchhoff node law • Kirchhoff loop law		

Examples

1. For the following systems find the mathematical model

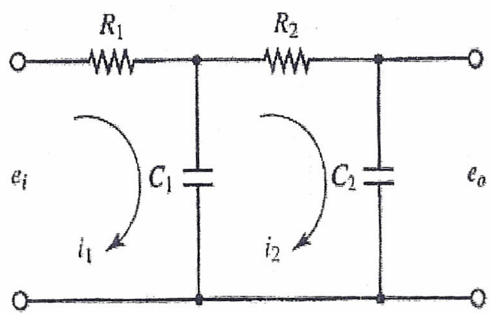


$$e_i = e_R + e_c$$

$$e_i = i * R + \frac{1}{c}\int_0^t i\,dt$$

$$e_c = \frac{1}{c}\int_0^t i\,dt$$

2. Write the mathematical model for the following system

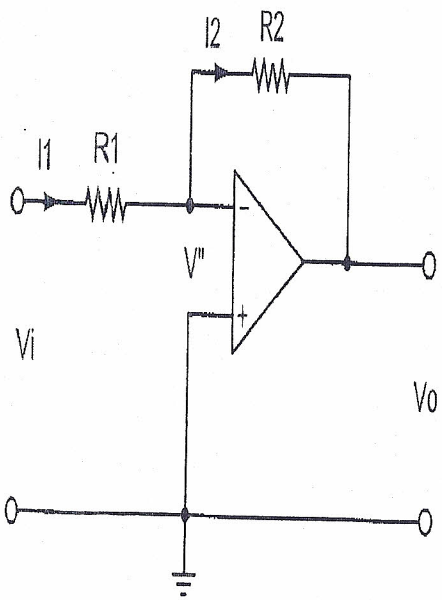


$$\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = e_i$$

$$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0$$

$$\frac{1}{C_2}\int i_2\,dt = e_o$$

3 . For the following inverting amplifier find the mathematical model.



The equation for this circuit can be obtained as follows: Define

$$i_1 = \frac{e_i - e'}{R_1}, \quad i_2 = \frac{e' - e_o}{R_2}$$

Since only a negligible current flows into the amplifier, the current i_1 must be equal to current i_2. Thus

$$\frac{e_i - e'}{R_1} = \frac{e' - e_o}{R_2}$$

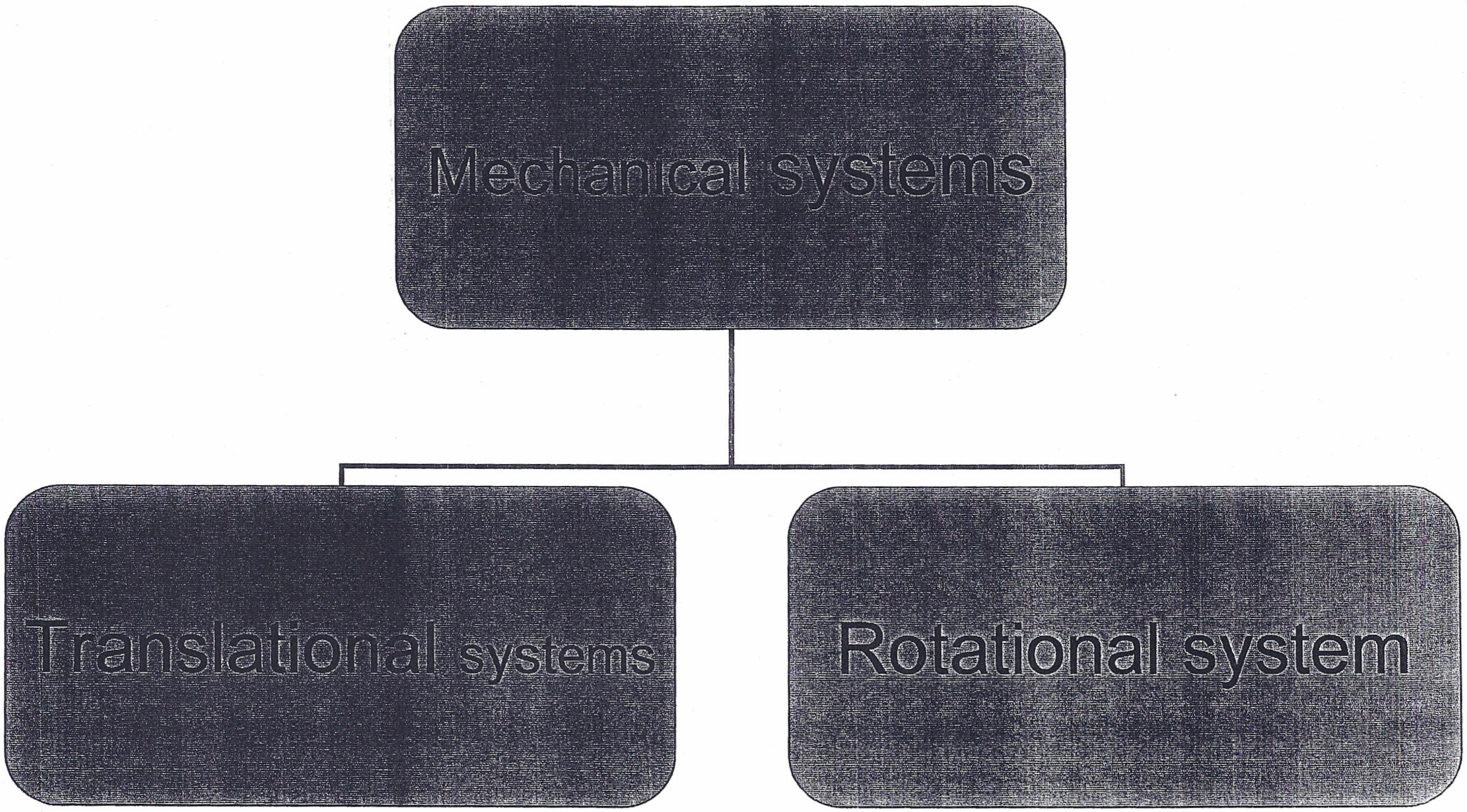
Since $K(0 - e') = e_0$ and $K \gg 1$, e' must be almost zero, or $e' \doteqdot 0$. Hence we have

$$\frac{e_i}{R_1} = \frac{-e_o}{R_2}$$

or

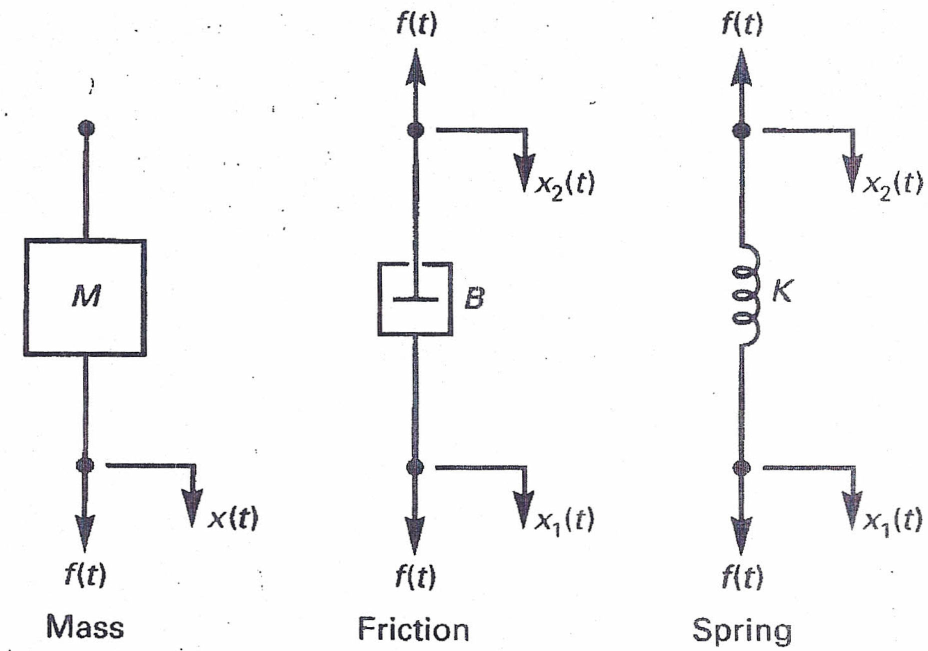
$$e_o = -\frac{R_2}{R_1} e_i$$

Thus the circuit shown is an inverting amplifier. If $R_1 = R_2$, then the op-amp circuit shown acts as a sign inverter.


Mechanical systems
Translational systems
Rotational system

Mechanical Translational system

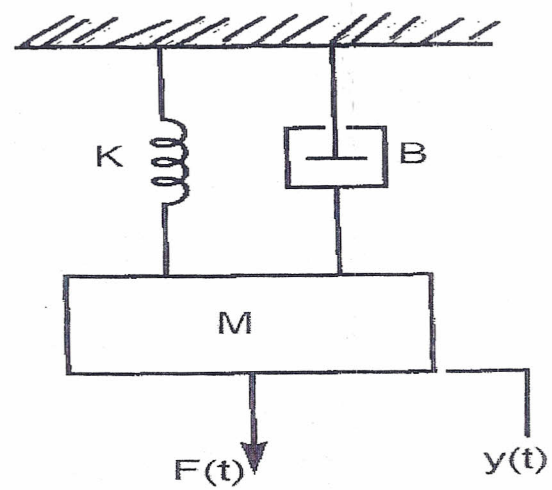
- The mechanical T.S elements involved are
  - Mass
  - Linear spring
  - Damping (friction)
- The basic laws are
  - Newton's law
  - Hooke's law



$$f(t) = Ma(t) = M\frac{dv(t)}{dt} = M\frac{d^2x(t)}{dt^2} \qquad f(t) = B\left[\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt}\right] \qquad f(t) = K\left[x_1(t) - x_2(t)\right]$$

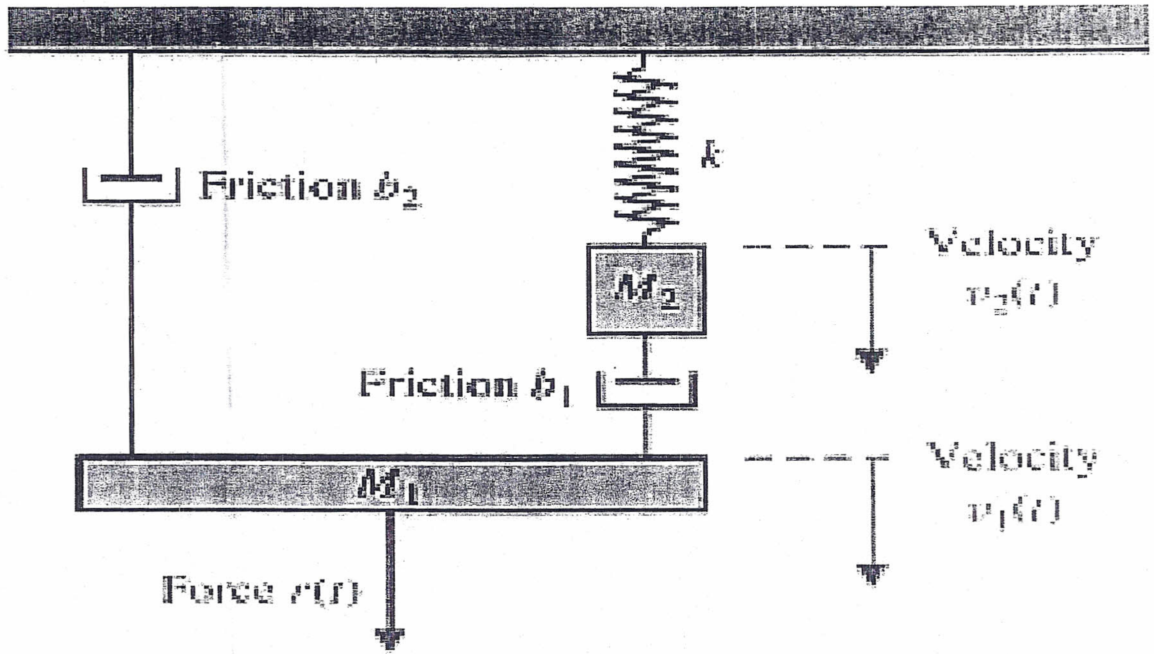
Examples

- For the following simple systems . Write the mathematical model.



$$\sum F = m\frac{d^2x}{dt^2} = f(t) - f_c - f_B$$

$$= f(t) - kx - B\frac{dx}{dt}$$

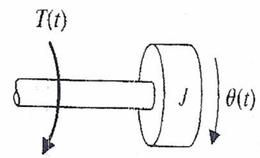


*For mass*1

*For mass*2 $$M_2 \frac{d^2 x_2}{dt^2} = b_1\left[\frac{dx_1}{dt} - \frac{dx_2}{dt}\right] - kx_2$$

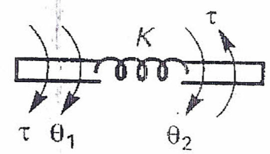
Mechanical Rotational system

- The mechanical Rotational system elements involved are
    - Moment of inertia



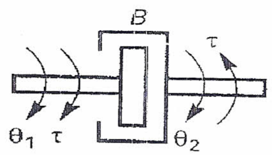
$$\tau(t) = J\frac{d^2\theta(t)}{dt^2} = J\frac{d\omega(t)}{dt}$$

- Where $\tau(t)$($T(t)$) is applied torque,
- J is moment of inertia
- θ is angle of rotation
    - Torsion spring
- K is the spring coefficient



$$\tau(t) = K[\theta_1(t) - \theta_2(t)]$$

- Damping (friction)
- B is the damping coefficient

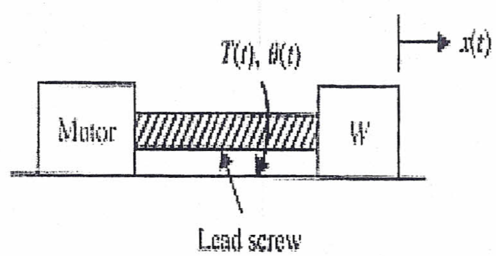


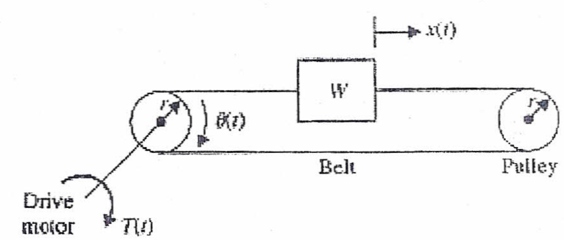
$$\tau(t) = B\left(\frac{d\theta_1(t)}{dt} - \frac{d\theta_2(t)}{dt}\right) = B[\omega_1(t) - \omega_2(t)]$$

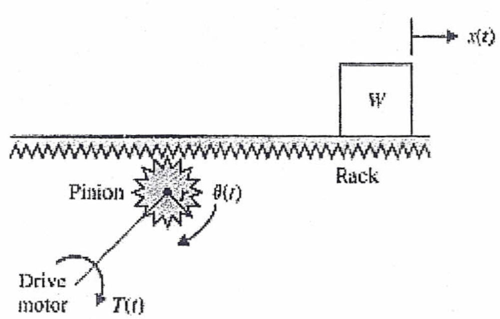
- The basic laws are
    - Newton's law
    - Hooke's law

Conversion between Translational and Rotational Motions

- In motion-control systems it is often necessary to convert rotational motion into translation . For example a load may be controlled to move along a straight line through a rotary motor-and screw assembly or through a rack-and-pinion or through a pulley by rotary motor . The systems given below can be represented by simple equivalent inertia
- Where L is defined as the linear distance that the mass travels per revolution of the screw





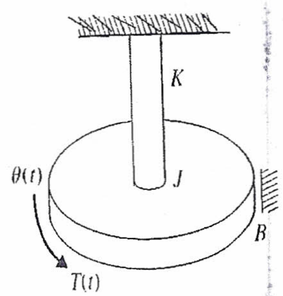


$$J = Mr^2 = \frac{W}{g} r^2$$

$$J = \frac{W}{g}\left(\frac{L}{2\pi}\right)^2$$

Examples

- For the following systems find the mathematical models



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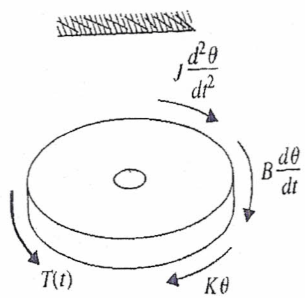
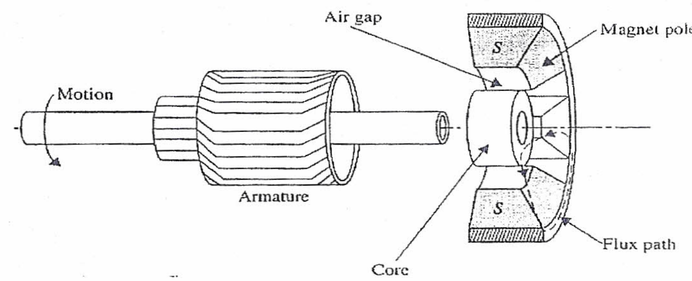


(b)

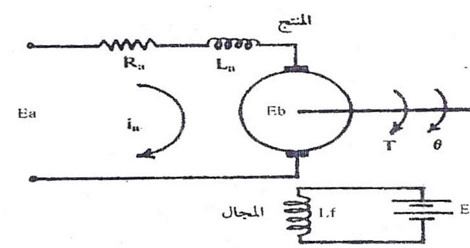
$$T(t) = J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} + k\theta(t)$$

Electro-mechanical systems

- The most common electro-mechanical system are
  - Dc motor



- Dc motor armature control (constant field current)



$$E_f = L_f \frac{di_f}{dt}$$

$$E_a = R_a i_a + L_a \frac{di_a}{dt} + E_b$$

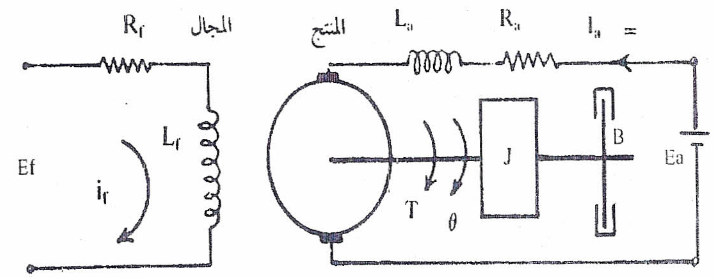
$$E_b = k_b \frac{d\theta}{dt}$$

$$Flux \quad \phi = \mathrm{k_f} * i_f$$

$$Torque \quad T_m \,\alpha\, \phi * i_a = k_m i_a$$

Apply Newton's second law of motion yields

$$\mathrm{T} = \mathrm{J}\frac{\mathrm{d}^2\theta}{dt^2} + B\frac{\mathrm{d}\theta}{dt}$$



- Dc motor field control (constant armature current)
  - Drive the mathematical model for this type.

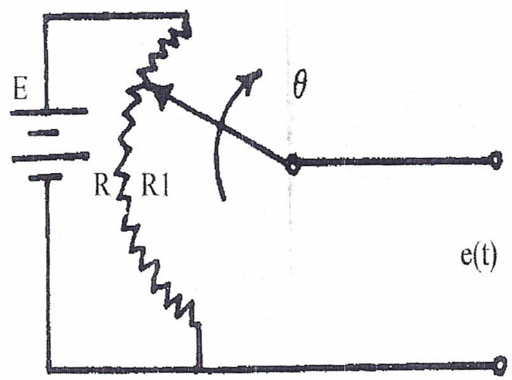
potentiometer

- A potentiometer is an electromechanical transducer that converts mechanical energy into electrical energy.
- The input to the devices is in the form of a mechanical displacement.
- When a voltage is applied across the fixed terminals of the potentiometer, the output voltage , which is measured across the variable terminal and ground, is proportional to input displacement.

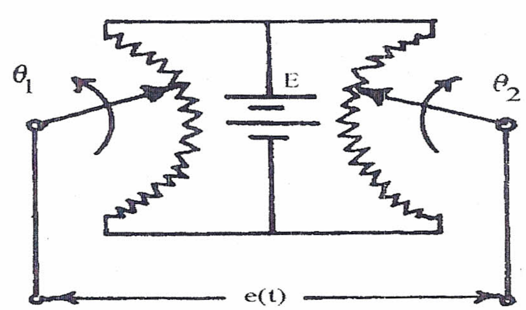
$$e(t) = \frac{R_1}{R} E = \frac{\theta(t)}{\theta_{max}} E$$

$$e(t) = k_\theta \theta(t)$$

$$where \qquad k = \frac{E}{2\pi N}, \qquad \text{N = no of turn}$$



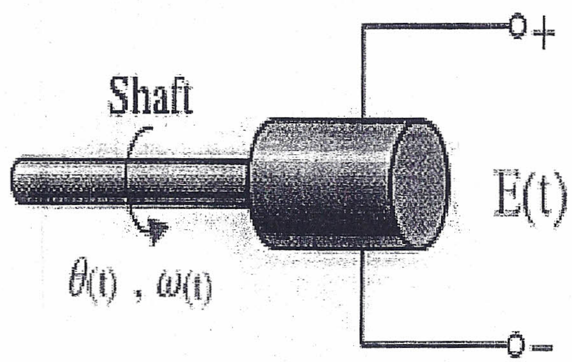
- For more flexible arrangement is using two potentiometers connected in parallel



$$e(t) = K_\theta[\theta_1(t) - \theta_2(t)]$$

Tachometers

- Tachometers is device to convert mechanical energy (speed) into electrical energy.
- The dynamics of the tachometer can be represented by the equation



$$e(t) = k\frac{d\theta(t)}{dt} = k\omega(t)$$

Gear system

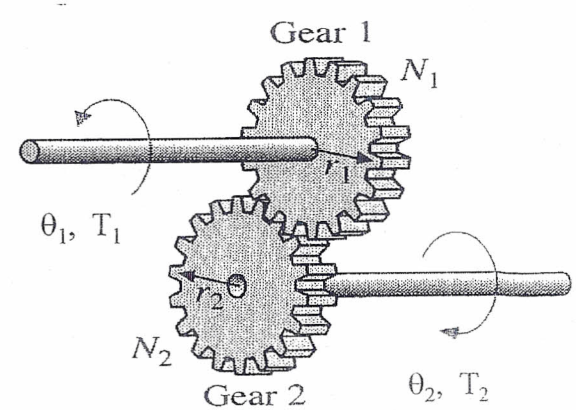
- The gear system can be modeled under the following conditions
  1. The gears are rigid.
  2. Gear teeth are meshing perfectly.
  3. Gears have no inertia.

- Then the distance traveled by one gear must be equal to that traveled by other, or

$$L = \theta_1 r_1 = \theta_2 r_2$$

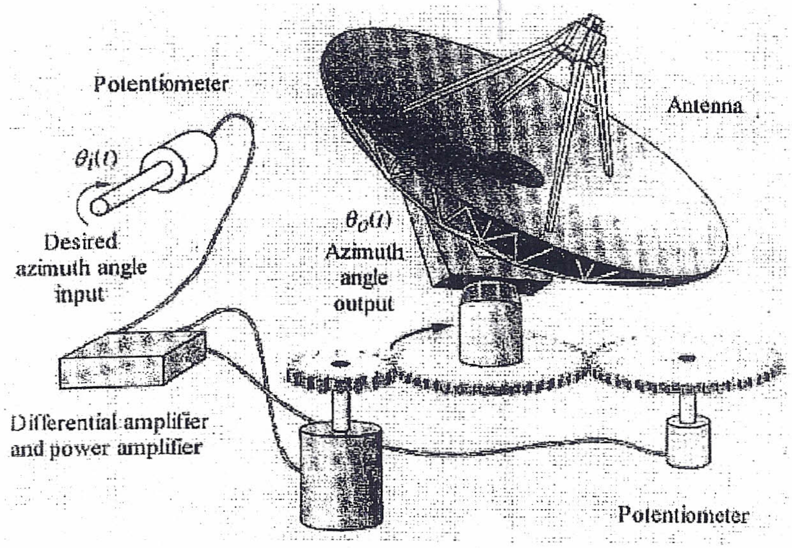
- The force exerted by one gear must equal the reaction force of the other at the point of contact

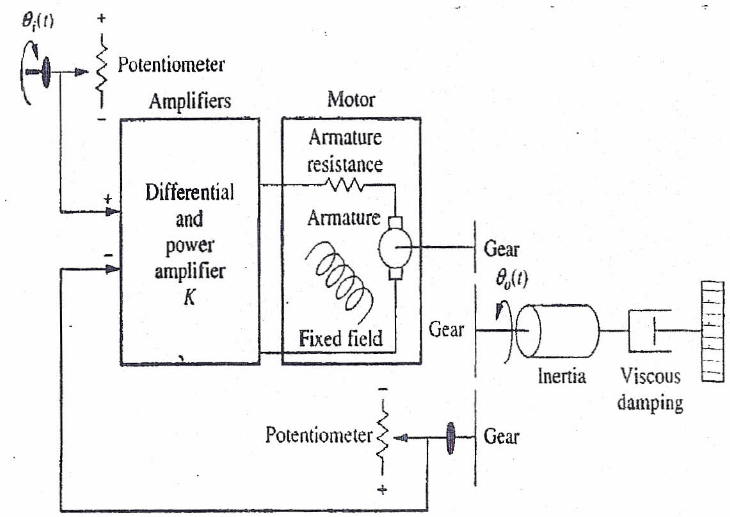
$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{r_1}{r_2}$$

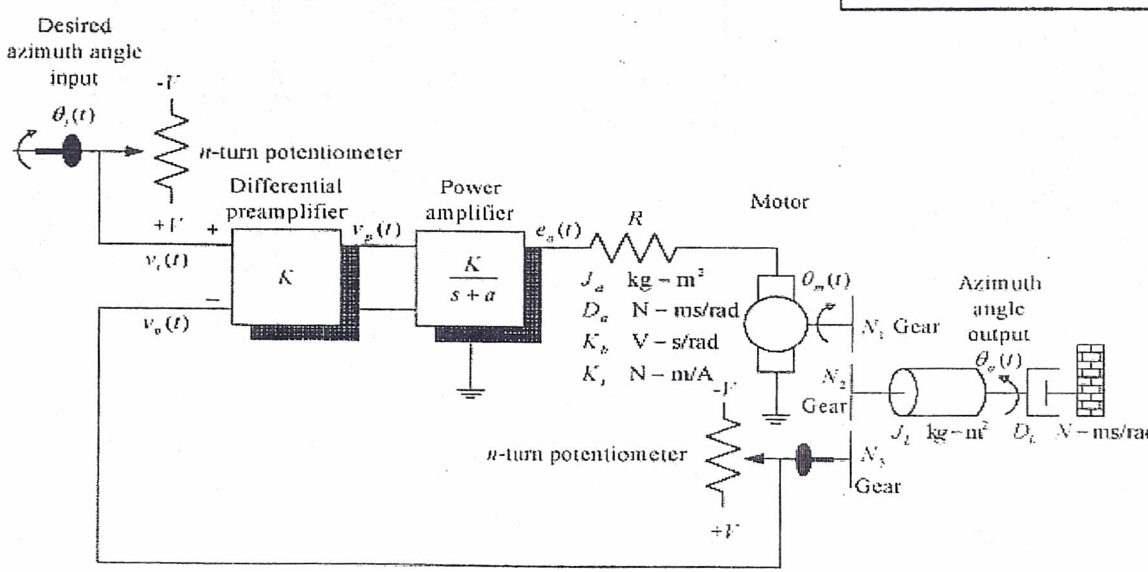


Antenna system

- For the following Antenna system find the schematic diagram

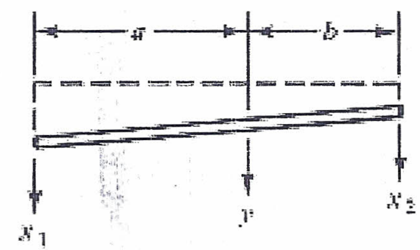






Lever mechanism

- Lever system is shown as



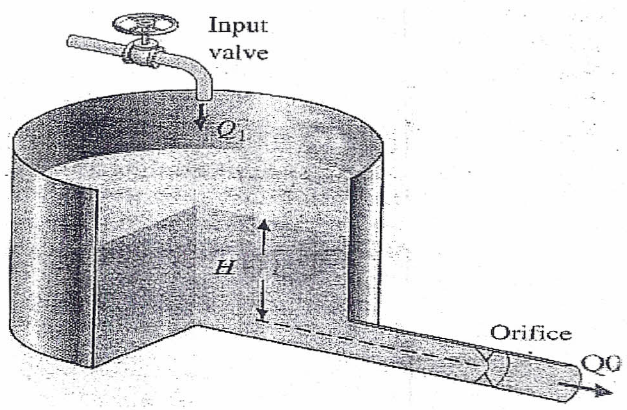
- For small angles from horizontal , the total motion y equal the sum of motion due to x1 with x2 = 0 and that due to x2 with x1 = 0.

$$y = \frac{b}{a+b} x_1 + \frac{a}{a+b} x_2$$

Liquid Level system

- Consider the system given below. In steady-state

  net volumetric in flow rate= change per unit time of the volume



$$\frac{dv}{dt} = Q_i - Q_o$$

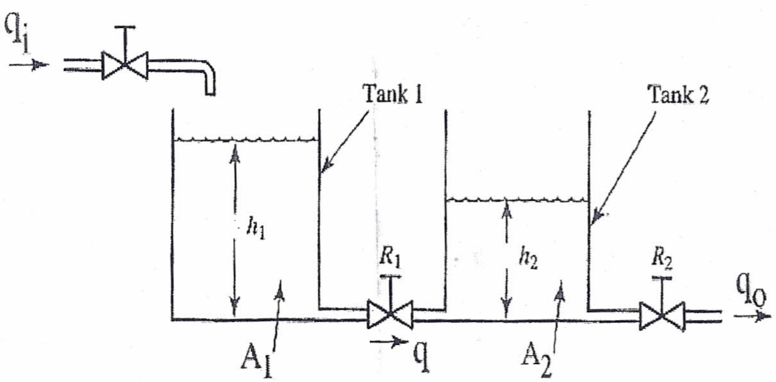
where v is tank volume

The relation between H and Qo is

$$Q_o = \frac{H}{R}$$

where R is Fluid resistance

- For two-tank liquid system

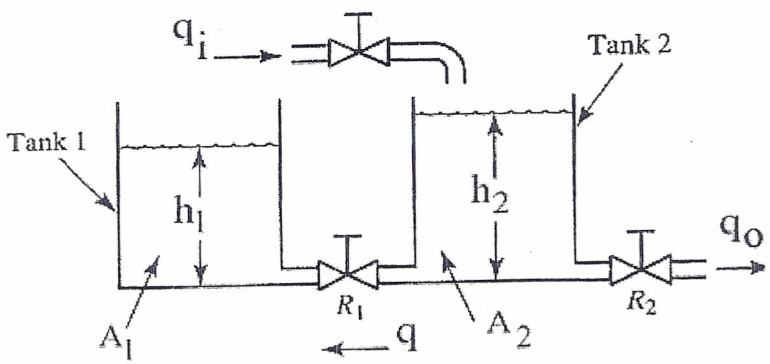


$$\frac{h_1 - h_2}{R_1} = q$$

$$A1\frac{dh_1}{dt} = q_i - q$$

$$\frac{h_2}{R_2} = q_o$$

$$A2\frac{dh_2}{dt} = q - q_o$$



$$\frac{h_2 - h_1}{R_1} = q$$

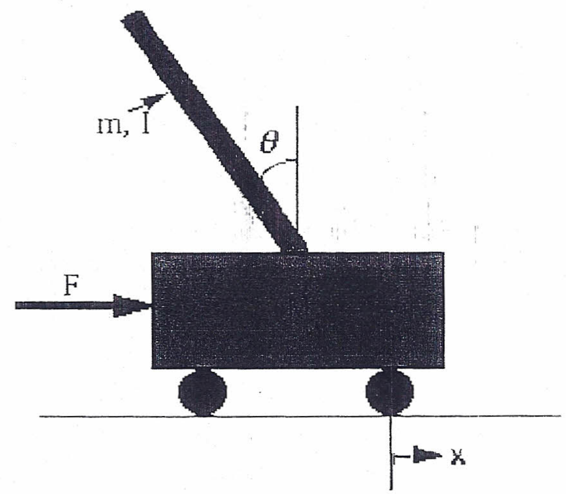
$$A1\frac{dh_1}{dt} = q$$

$$\frac{h_2}{R_2} = q_o$$

$$A2\frac{dh_2}{dt} = q_i - q - q_o$$

Modeling an Inverted Pendulum

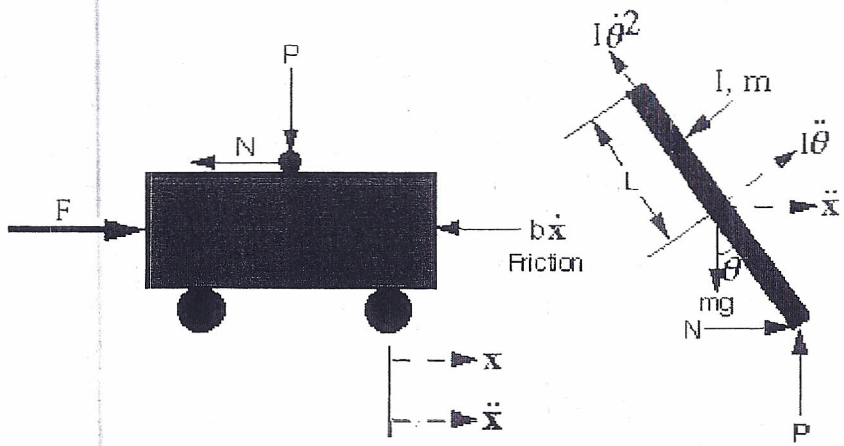
- The schematic diagram of inverted Pendulum is given as



- let's assume that
- M mass of the cart = 0.5 kg
- m mass of the pendulum = 0.2 kg
- b friction of the cart = 0.1 N/m/sec
- L length to pendulum center of mass = 0.3 m
- I inertia of the pendulum = 0.006 kg*m^2
- F force applied to the cart
- x cart position coordinate
- theta pendulum angle from vertical

Force analysis and system equations

- Below are the two Free Body Diagrams of the system.



- Summing the forces in the Free Body Diagram of the cart in the horizontal direction, the following equation of motion is obtained:

$$M\ddot{x}+b\dot{x}+N=F \quad (1)$$

- Note that you could also sum the forces in the vertical direction, but no useful information would be gained.
- Summing the forces in the Free Body Diagram of the pendulum in the horizontal direction, an equation for N is :

$$N=m\ddot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta$$

These set of equations should be linearized about theta = Pi.

- Assume that theta = Pi + φ (φ represents a small angle from the vertical upward direction). Therefore,

  cos(theta) = -1, sin(theta) = -φ , and (d(theta)/dt)^2 = 0.

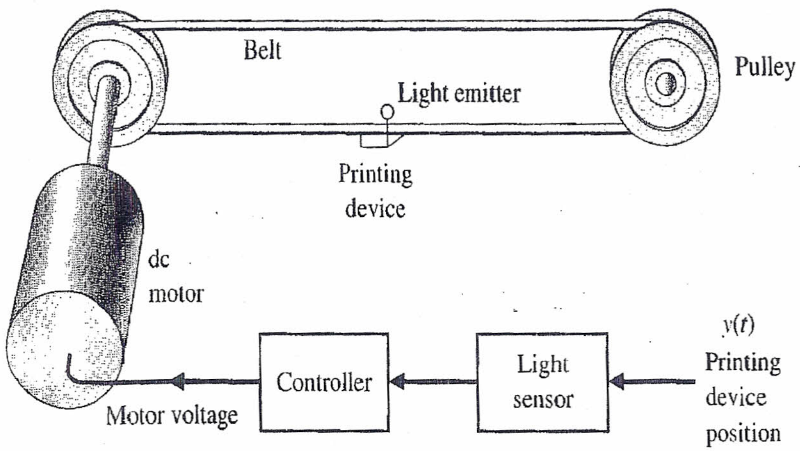
- After linearization the two equations of motion become (where u represents the input):

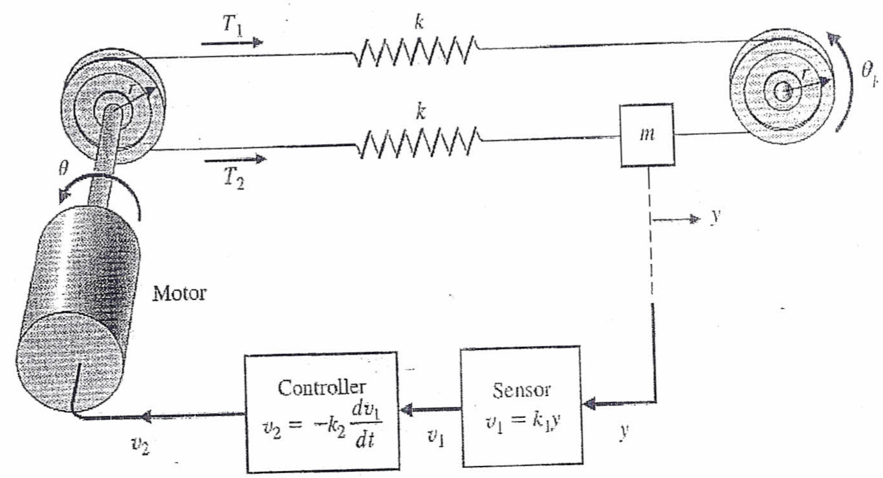
$$\left(I + ml^2\right)\ddot{\phi} - mgl\phi = ml\ddot{x}$$

$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u$$

Model a belt drive printer

- A schematic diagram of a belt drive printer is shown as





- In this model , a light sensor is used to measure the position of the printing device.
- The belt tension adjusts the spring flexibility of the belt.
- Where

$$v_1 = k_1 y, \qquad v_2 = -\left[k_2 \frac{dv_1}{dt}\right]$$

System model

- The relationship between the pully rotation (θp) and printing device displacement y is

$$y = r\theta_p$$

- The tension T1 is $$T_1 = k(r\theta - r\theta_p) = k(r\theta - y)$$

- The tension T2 is $$T_2 = k(y - r\theta)$$

- Then the total net tension is $$T_1 - T_2 = m\frac{d^2y}{dt^2} = 2k(r\theta - y)$$

- In motor side (assume L =0) then
- The field current is

$$i_f = \frac{v_2}{R}$$

- Motor torque Tm is

$$T_m = k_m i_f = k_m \frac{v_2}{R}$$

- The motor torque provides the torque to drive the belts plus the disturbance or undesired load torque, so that Tm = T + Td
- Where
- $$T = J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + r(T_1 - T_2)$$

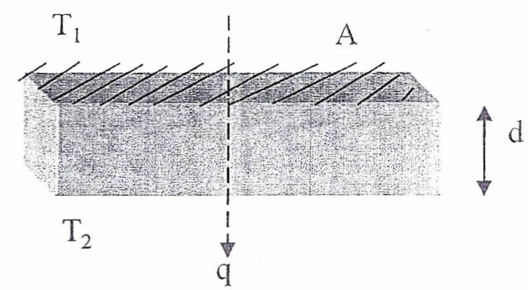
Thermal system

- Thermal system consists from

1. Thermal resistance (Rt)

  - A wall of area A separates regions with temperature T1 and T2
  - The heat flow rate q, in units of heat per unit of time, is proportional to the temperature difference T1-T2 and to the area A and flows toward the lowest temperature .
  - The constant of proportionality is the heat transfer coefficient h .

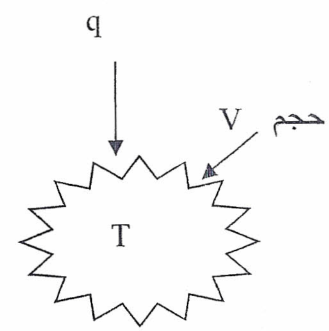
$$q \propto \frac{A}{d}(T_1 - T_2)$$



$$q = \frac{kA}{d}(T_1 - T_2) = hA(T_1 - T_2) = \frac{T_1 - T_2}{R_t}$$

2- Thermal Capacitance

- let q be the net heat flow rate into a volume V of a material with mass density ρ and specific heat c (= heat required to raise the temperature of a unit mass by 1°) .



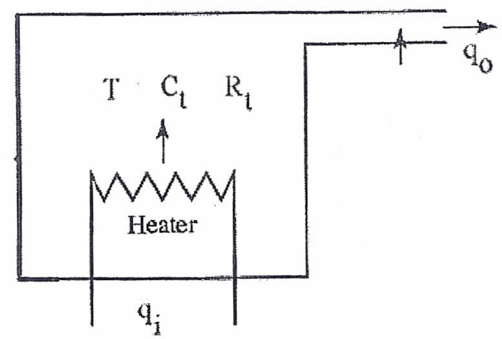
- The inflow q of heat per second must equal the change per second of heat stored in volume V.

$$q = \rho V c \dot{T} = C_t \dot{T}$$

- Where C_t is called a thermal capacitance.

Examples

- For the following thermal system find the mathematical model.



- The behavior of the system can be modeled as

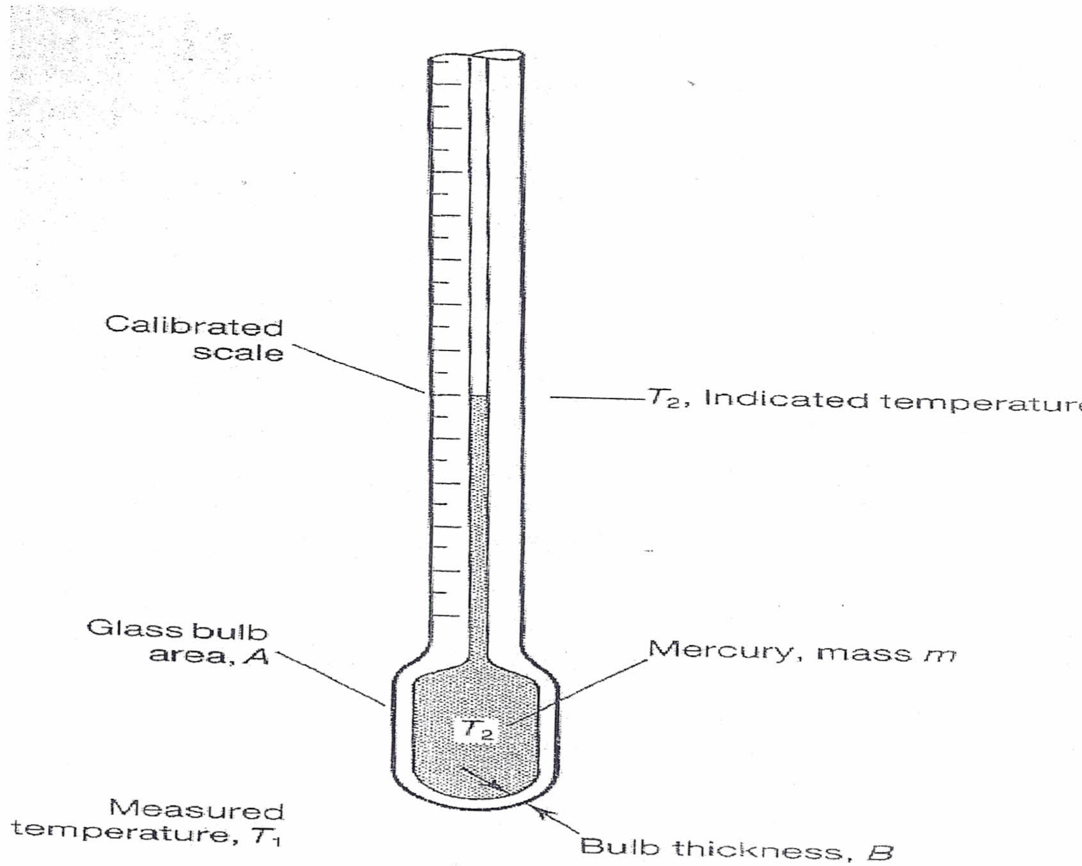
1. The net flow is $$q_i - q_o = C_t \frac{dT}{dt}$$
2. The output heat is $$q_o = \frac{T}{R_t}$$
3. Therefore the mathematical model is $$R_t q_i = T + R_t C_t \frac{dT}{dt}$$

Model a liquid-in-glass thermometer

To obtain a linear model relating the thermometer indicated temperature, T2, to the measured temperature, T1

According to Fourier's law, the heat flow rate through the glass bulb is proportional to the temperature gradient within the bulb,

$$q = kA\frac{T1-T2}{B}$$



Where k is thermal conductivity of glass bulb and A its surface area.

The temperature difference across the glass bulb will be (T1-T2), B bulb thickness

$$\frac{dT}{dx} = \left(\frac{T1 - T2}{B} \right)$$

$$q = kA \frac{T1 - T2}{B}$$

Net heat flow = Rate of change of internal heat $$q = \frac{dH}{dt}$$

where H is the internal heat of the mass m,of mercury at uniform temperature, T2 is given by

H =m cp T2

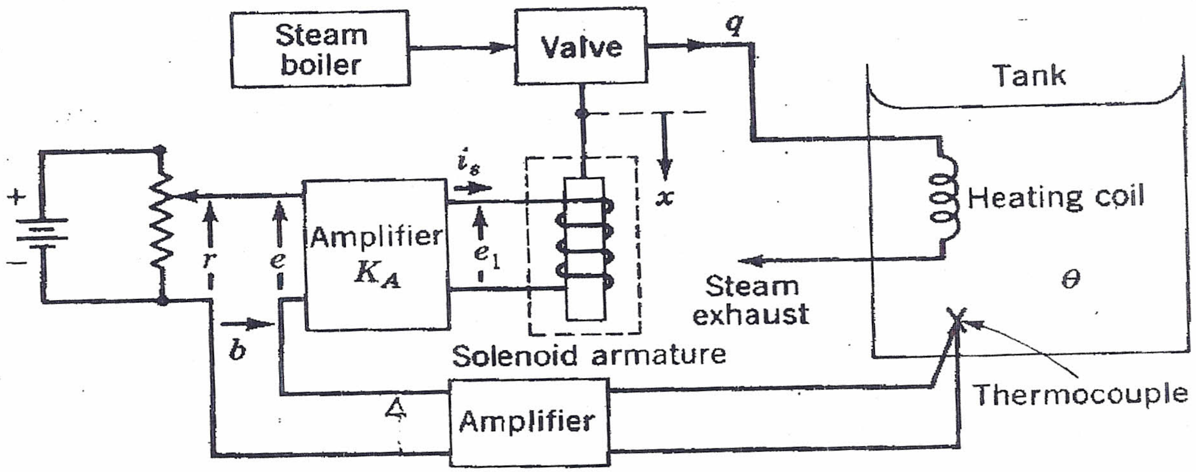
where cp is specific heat.

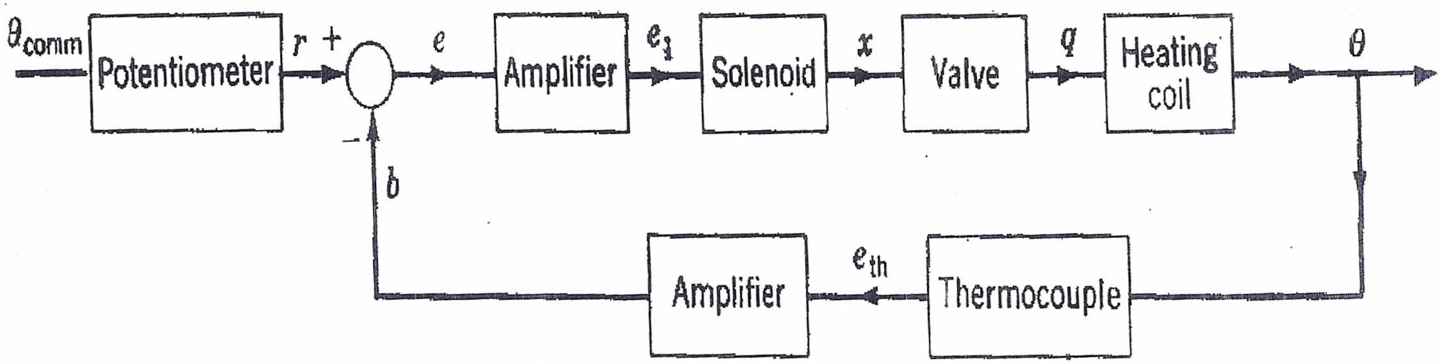
Finally

$$kAT_2 + mCpB \frac{dT_2}{dt} = kAT_1$$

A temperature-control system

- This example is an industrial –process control . Draw the block diagram





Fluid system

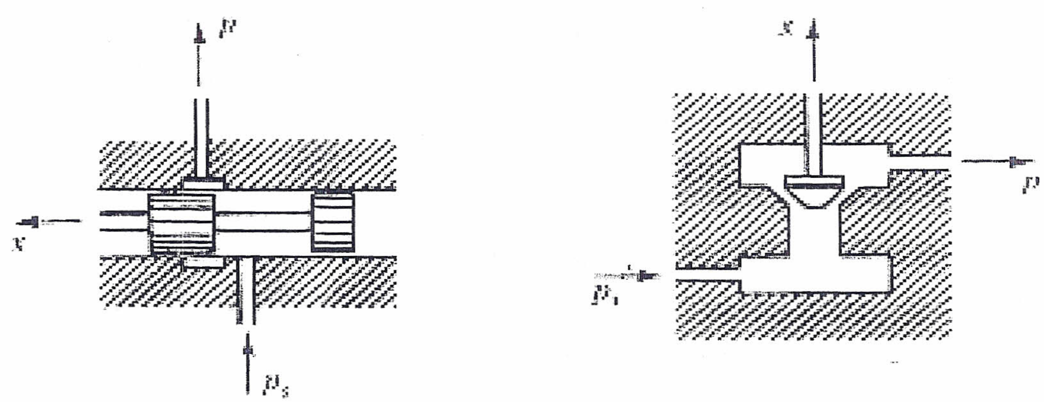
- Fluid system elements are defined as

1. **Control valves**

Two common types of valves are shown below

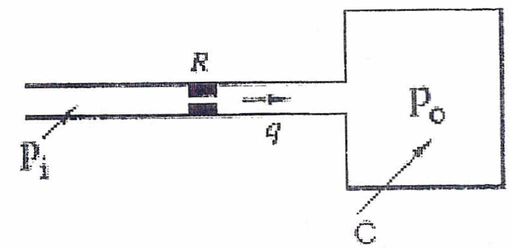
- The supply pressure is p_s.
- The output pressure is p.
- For x=0 the output port is just blocked off.
- Valve flow increases with both x and valve pressure drop (ps – p).

$$q = k_x x + k_p (p_s - p) = k_x x + k_p p_d$$



(a) (b)

2- **Pneumatic tank**



- The flow q through R (Fluid resistance) can be written as

$$q = \frac{p_i - p_o}{R}$$

- Due to this flow the tank pressure will raise according to

$$q = C\frac{dp_o}{dt}$$

- Rearrangement the above equations yields

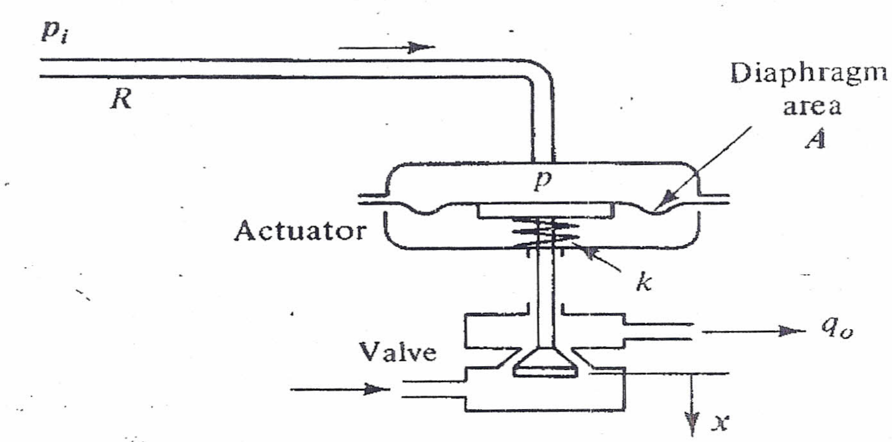
$$RC\frac{dp_o}{dt} + p_o = p_i$$

- So that the transfer function is

$$\frac{p_o}{p_i} = \frac{1}{RCs + 1}$$

3- Pneumatically actuated valve

- Pneumatically actuated valve is used extensively in level control systems.
- The block diagram is given as



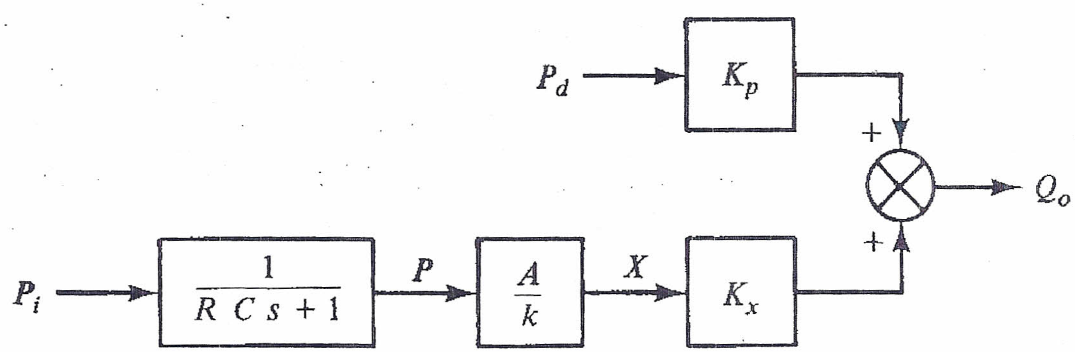
- The relationship between control pressure **pi** to pressure **p** is represented by

$$q = \frac{p_i - p}{R}$$

- Due to this flow the pressure will raise according to $$q = C\frac{dp}{dt}$$
- Diaphragm motion x is to small that the capacitance C of the space above it is about constant

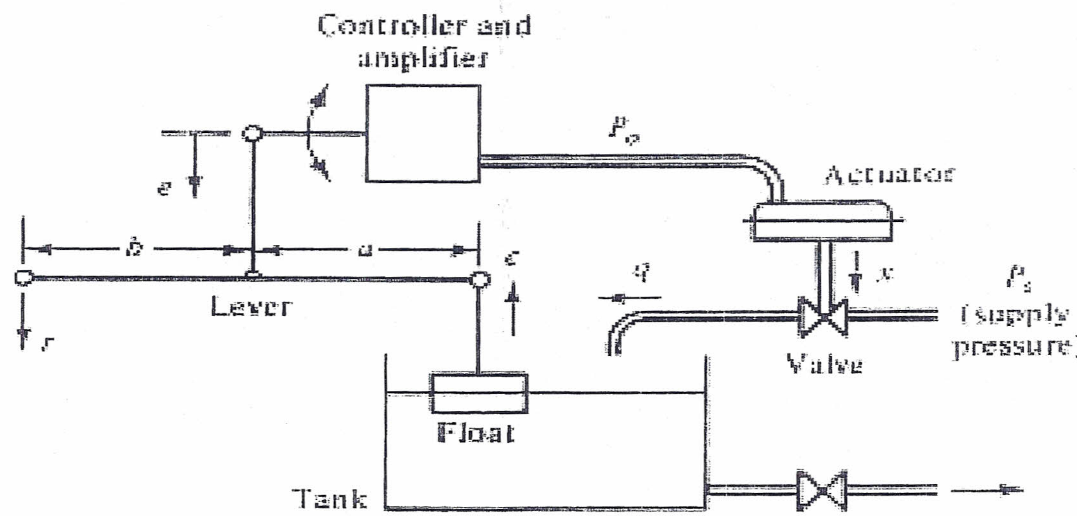
- The downward pressure force (Ap) on the diaphragm must be counterbalanced by the spring force (kx)
- $$Ap = kx$$
- the model for the valve flow q_o is given by $$q_o = k_x x + k_p pd$$
- Where pd is the pressure drop across the valve.
- Finally ,the following transfer functions can be derived

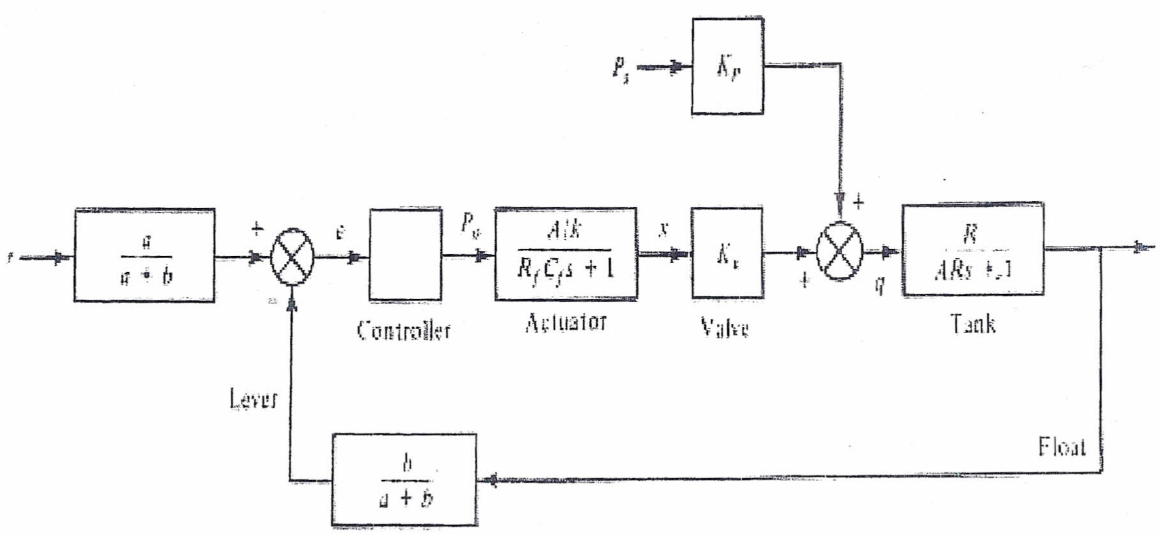
$$\frac{p(s)}{p_i(s)} = \frac{1}{RCs+1}, \qquad \frac{Q_o(s)}{p_i(s)} = \frac{Ak_x/k}{RCs+1}$$



problem

- Find the mathematical model of





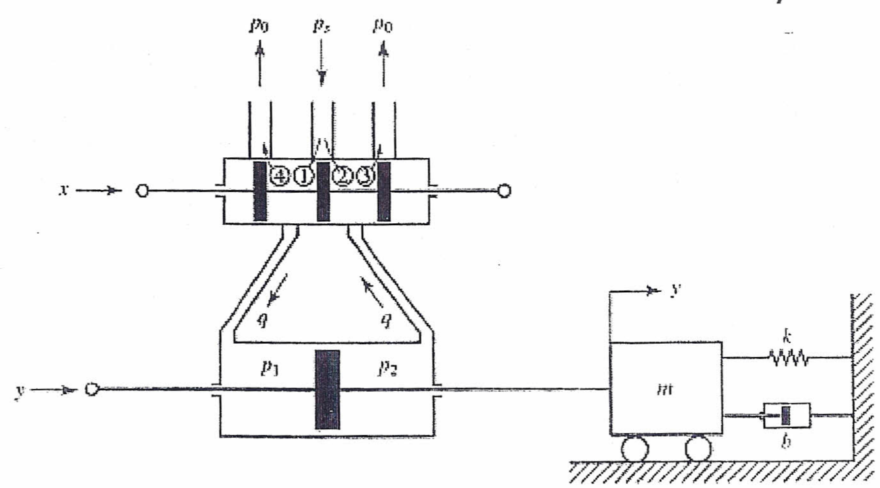
- If a load consists of mass (m), spring (k) and damper (b) is connected to the hydraulic cylinder as shown below. Then
- the pressure on piston is p, and the force balance equation is

$$pA = m\ddot{y} + b\dot{y} + ky$$

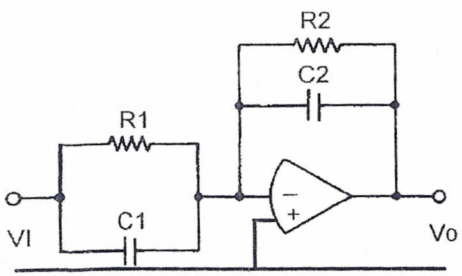
- The valve flow is $q = k_x x + k_p(p_s - p)$
- The flow q to the cylinder with compressibility flow associated with pressure variations is

$$q = A\dot{y} + \frac{V}{\beta}\dot{p}$$

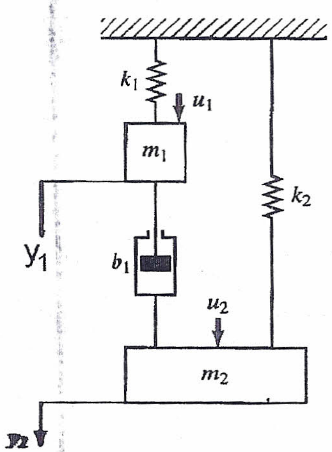
- Where V is the volume under pressure, and compressibility flow is $\frac{V}{\beta}\dot{p}$

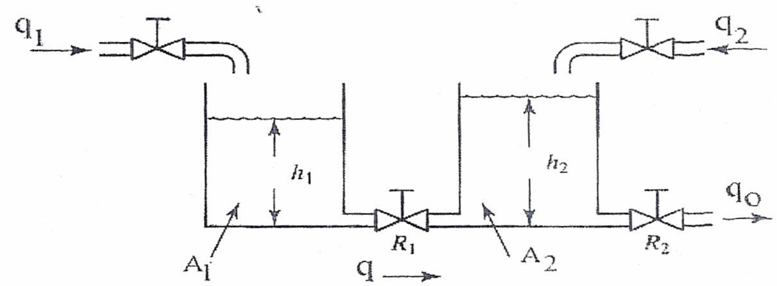


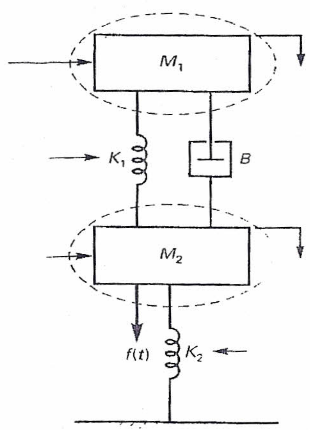
Problems

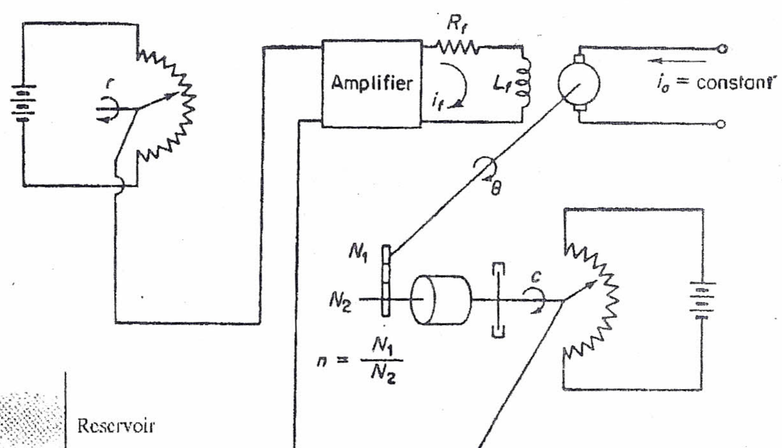

R2
C2
R1
C1
VI
Vo

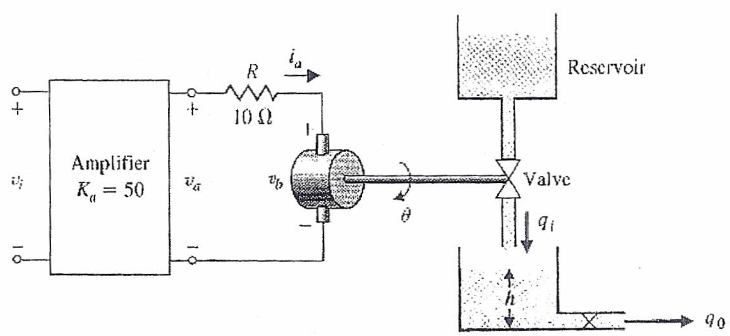
e


k_1
u_1
m_1
k_2
y_1
b_1
u_2
m_2
y_2


q_1
q_2
h_1
h_2
q_o
R_1
R_2
A_1
A_2
q


M_1
K_1
B
M_2
f(t)
K_2


R_f
Amplifier
L_f
i_f
i_o = constant
θ
N_1
N_2
c
$n = \frac{N_1}{N_2}$


Amplifier
$K_a = 50$
v_i
v_a
v_b
R
10 Ω
i_a
θ
Reservoir
Valve
q_i
h
q_o

Linearization of non-linear system

- The linearization procedure is based on the expansion of the nonlinear function into a **Taylor series** about the operating point and this can be done by assume that the variable deviate only slightly from some operating condition i.e neglect higher-order terms of Taylor series.

Steps for linearization

1. Define the nonlinear element.
2. Write the nonlinear equation which represents the nonlinear element.
3. Assume that the variable deviate only slightly from some operating condition.
4. Approximate the the nonlinear element by using **Taylor series** .

Let the input – output model for a static element be written as

$$y = f(x)$$

If the normal operating condition corresponds to xo, yo then

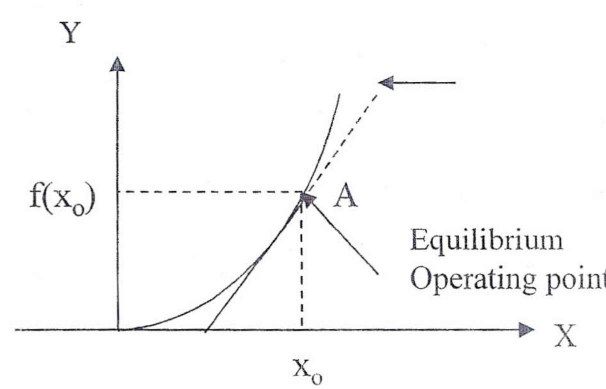
$$f(x) = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x=x_0} \frac{(x-x_0)}{1!} + \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_0} \frac{(x-x_0)^2}{2!} + \ldots\ldots$$

if the variation x – xo is small , then the above equation can be written as

$$y = f(x) = f(xo) + K\,(x-xo)$$

Where

$$k = \left.\frac{df}{dx}\right|_{x=xo}$$

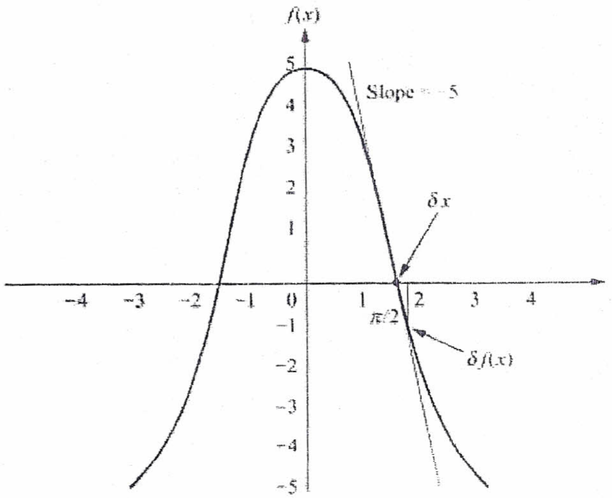


Then the linearized equation can be written as

$$y - yo = k\,(x - xo)$$

Example 1 find the linearized model for the following system about the operating point

$$f(x) = 5\cos x \qquad x_0 = \frac{\pi}{2}$$



$$k = \left.\frac{\partial f}{\partial x}\right|_{x = x_o} = -5\sin(x)\Big|_{x=xo=\frac{\pi}{2}} = -5$$

$$f(x_o) = 5\cos(\frac{\pi}{2}) = 0$$

$$y - y(0) \cong k(x - x_0)$$

$$f(x) - 0 \cong -5\left(x - \pi/2\right)$$

$$f(x) \cong -5\left(x - \pi/2\right)$$

- Example 2 $$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+\cos x=0$$ $$x=\pi/4$$

- Nonlinear portion $$f(x)=\cos x$$

$$k=\left.\frac{\partial f}{\partial x}\right|_{x=xo=\frac{\pi}{4}}=-\sin x=-\frac{\sqrt{2}}{2}$$

$$f(x_0)=\cos x|_{x=\frac{\pi}{4}}=\frac{\sqrt{2}}{2}$$

$$x-x_0=x-\pi/4=\delta x$$

$$x=\delta x+\pi/4$$

$$f(x)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\delta x$$

- Linear model $$\frac{d^2\left(\delta x+\pi/4\right)}{dt^2}+2\frac{d\left(\delta x+\pi/4\right)}{dt}+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\delta x=0$$

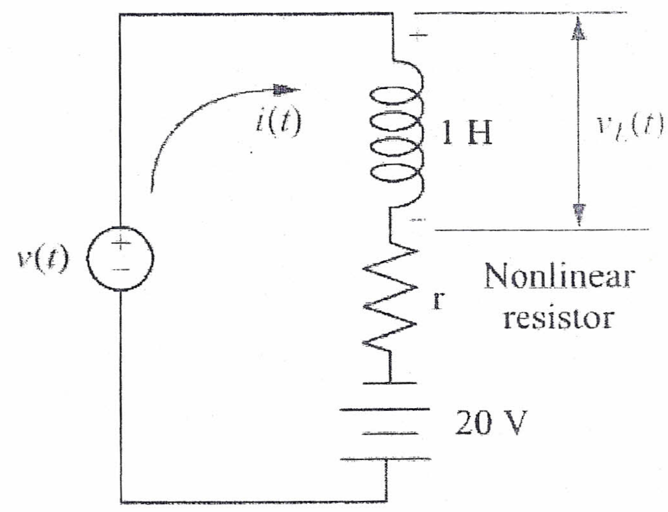
$$\frac{d^2\delta x}{dt^2}+2\frac{d\delta x}{dt}-\frac{\sqrt{2}}{2}\delta x=-\frac{\sqrt{2}}{2}$$

- Example 3
- Find the transfer function $V_L(s)/V(s)$ For the electrical network shown in figure below, which contains a nonlinear resistor whose voltage-current relationship is defined by

$$i = i_r = 2e^{0.1V_r}$$

- Solution
- Use kirchhoff's voltage law to sum the voltage in the loop to obtain the nonlinear differential equation.
- The voltage across the non-linear resistor is $V_r = 10\ln\left(\frac{i_r}{2}\right)$
- The current-voltage equation around the loop is

$$L\frac{di_r}{dt} + 10\ln\left(\frac{i_r}{2}\right) - 20 = v(t)$$



- Evaluate the equilibrium solution, by set the source
- Voltage signal equal zero i.e
- with v(t) = 0, the circuit consists of a 20 V battery in
- Series with the inductor and non-linear resistor.
- In the steady-state, the voltage across the inductor will
- be zero, and a given battery voltage will be cross the
- resistor, then
-

$$i_o = i_r = 2e^{0.1V_r} = 2e^{0.1*20} = 14.78$$

- This current is the equilibrium value, and the total current is

$$i = i_o + \delta i$$

- Then the non-linear part is $\ln\left(\frac{i}{2}\right) = \ln\left(\frac{i_0 + \delta i}{2}\right)$
- Linearize this part yields

$$\ln\left(\frac{i_o + \delta i}{2}\right) - \ln\left(\frac{i_o}{2}\right) = \left.\frac{d(\ln\left(\frac{i}{2}\right))}{di}\right|_{i=i_o} \delta i = \frac{1}{i_o}\delta i$$

- Or $\ln\left(\frac{i_o + \delta i}{2}\right) = \ln\left(\frac{i_o}{2}\right) + \frac{1}{i_o}\delta i$
- The linearized equation then is $L\frac{d\delta i}{dt} + 10\left(\left(\ln\left(\frac{i_o}{2}\right) + \frac{1}{i_o}\delta i\right)\right) - 20 = v(t)$
- Letting L = 1 and io = 14.78 yields $\frac{d\delta i}{dt} + 0.677\delta i = v(t)$
- Taking the Laplace transform with zero initial conditions yields $\delta i(s) = \frac{V(s)}{s + 0.677}$
- The voltage across the inductor about the equilibrium point is $V_L(t) = L\frac{d}{dt}(i_o + \delta i) = L\frac{d\delta i}{dt}$
- Taking the Laplace transform,

$$V_L(s) = Ls\delta i(s) = s\delta i(s) = \frac{sV(s)}{s + 0.677}$$

- Then, the final transfer function is

$$\frac{V_L(s)}{V(s)} = \frac{s}{s + 0.677}$$

- # Multi-variable
- The function is $Y = f(x_1, x_2)$
- Taylor series expansion is

$$f(x_1,x_2) \cong f(x_1(0),x_2(0)) + \left.\frac{\partial f}{\partial x_1}\right|_{x_1 = x_1(0)} (x_1 - x_1(0)) + \left.\frac{\partial f}{\partial x_2}\right|_{x_2 = x_2(0)} (x_2 - x_2(0))$$

- Example consider the perfect gas low $P = \dfrac{mRT}{V}$

$$P \cong P_0 + \left.\frac{\partial P}{\partial T}\right|_{\substack{T=T_0 \\ V=V_0}} (T - T_0) + \left.\frac{\partial P}{\partial V}\right|_{\substack{T=T_0 \\ V=V_0}} (V - V_0) \qquad P_0 = \frac{mRT_0}{V_0}$$

$$\left.\frac{\partial P}{\partial T}\right|_{\substack{T=T_0 \\ V=V_0}} = \left.\frac{mR}{V}\right|_{\substack{T=T_0 \\ V=V_0}} = \frac{mR}{V_0} \qquad \left.\frac{\partial P}{\partial V}\right|_{\substack{T=T_0 \\ V=V_0}} = -\frac{mRT_0}{V_0}$$

$$\delta p = P - P_0, \delta_T = T - T_0, \delta_V = V - V_0$$

$$\delta_p \cong \frac{mR}{V_o}\delta_T + \frac{mRT_o}{V_o^2}\delta V$$

Problems

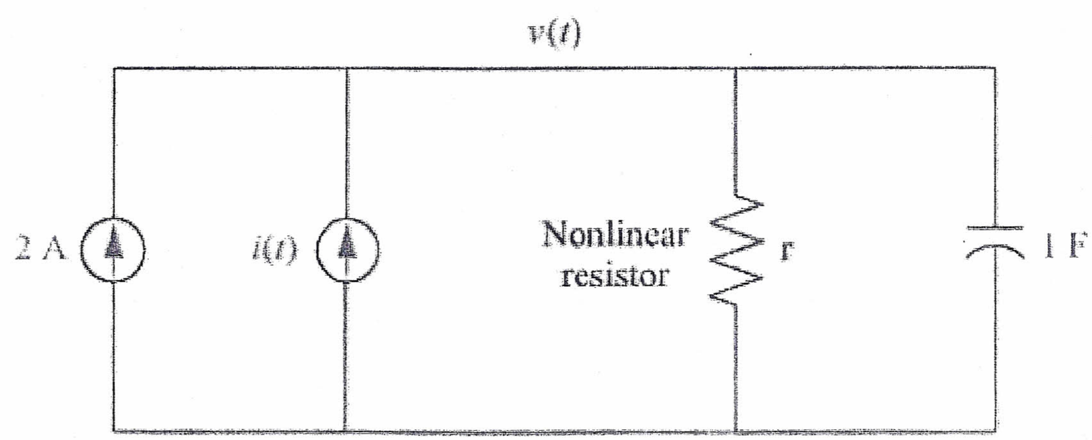
(1) $$\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2X=\sin x \qquad x=0 \quad x=\pi$$

(2) $$Q=KA\sqrt{\frac{2g}{p}(p_1-p_2)}$$ Where anp1-p2are variabAs

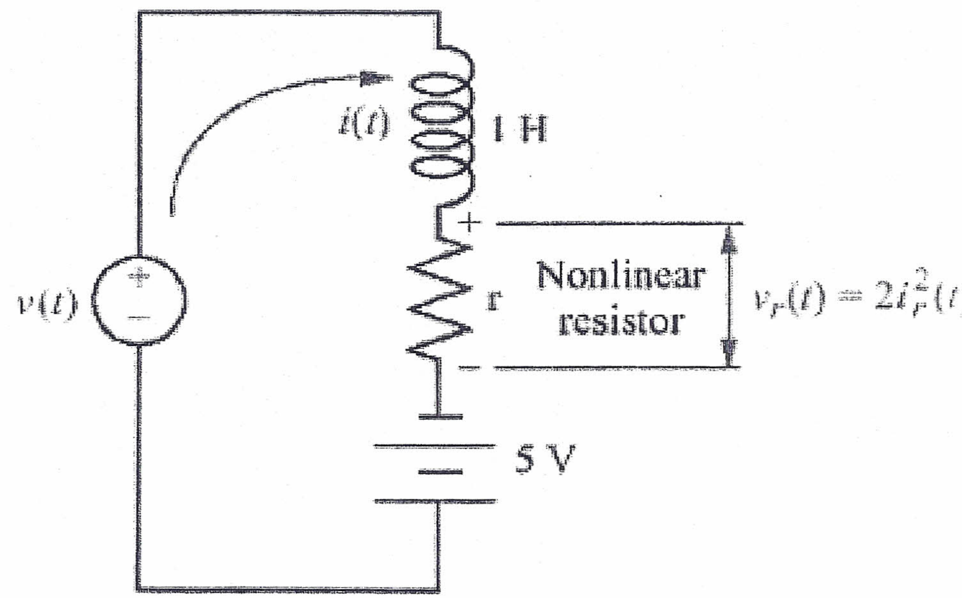
(3) Linearized the following system
$$\dot{x}_1 = x_2^2 + x_1\cos x_2$$
$$\dot{x}_2 = x_2 + (x_1+1)x_1 + x_1\sin x_2 \qquad x=0$$

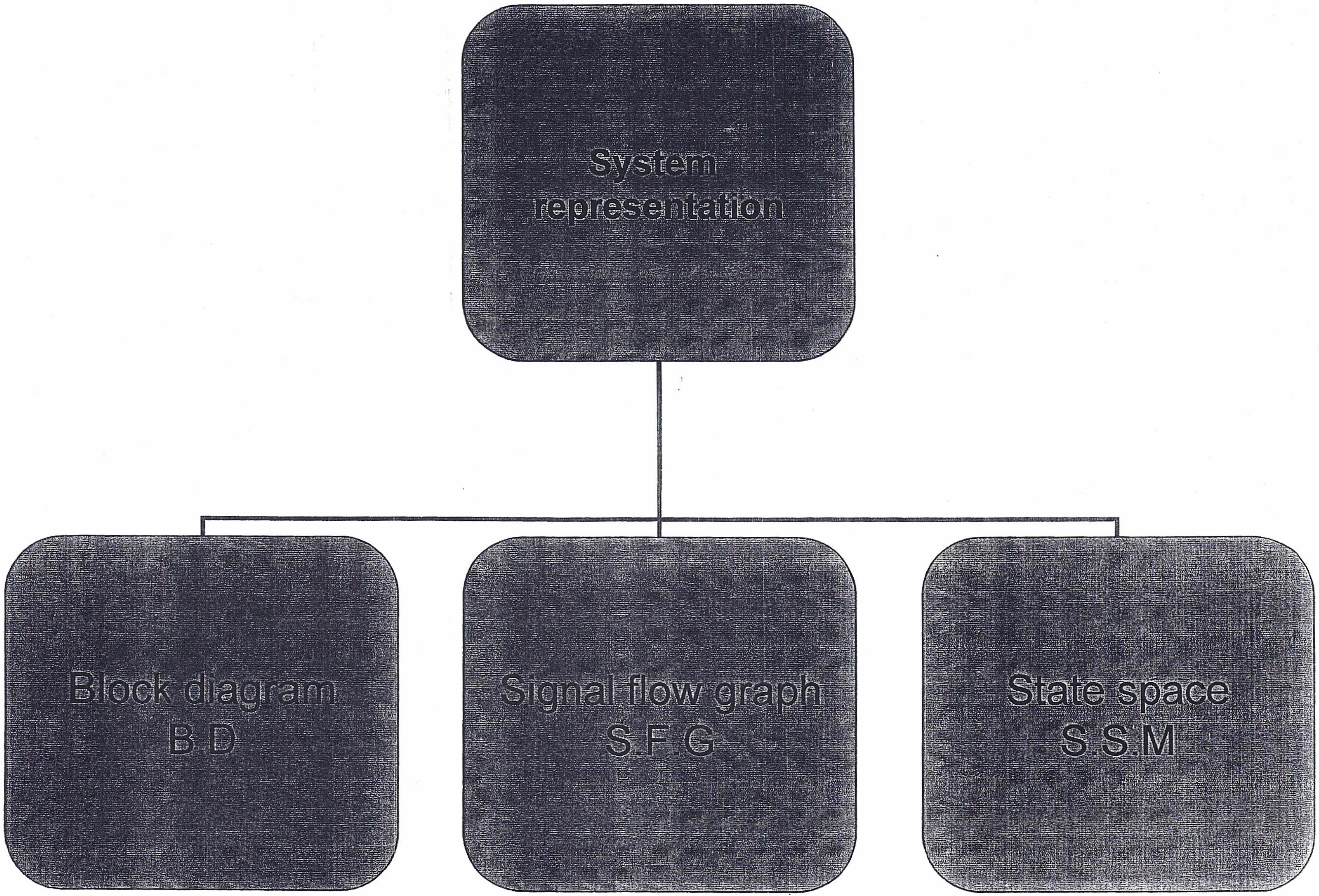
(5) Linearized the following system
$$\ddot{x} + 4\dot{x}^5 + (x^2+1)u = 0 \qquad x=0$$

(6) Find the linearized transfer function for the electrical network shown below, if $i_r = e^{V_r}$



(7) Find the linearized transfer function for the electrical network shown below, if




System representation
Block diagram
B.D
Signal flow graph
S.F.G
State space
S.S.M

1- Block diagram

- Block diagram: is a representation of the function performed by each component and of the flow of signals. Such a diagram depict the interrelationships which exist between the various components.
- In a block diagram, all system variables are linked to each other through functional blocks.
- Functional block is a symbol for the mathematical operation on the input signal to the block which produces the output.
- Transfer function: (T.F)
  - This phrase is very often used to characterize the input-output relationships of linear time-invariant systems and it is defined to be the ratio of the Laplace transform of the output to Laplace transform of the input under the assumption that all initial conditions are zero.

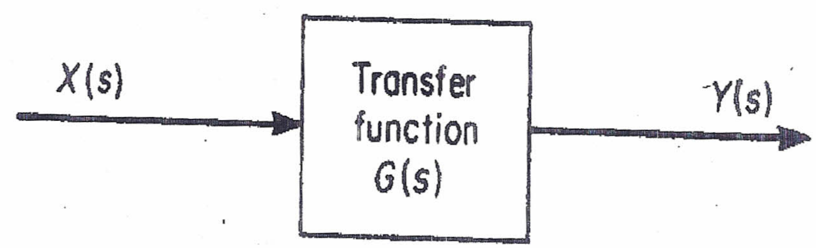
Consider the linear time-invariant system defined by the following differential equation:

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

where y is the output of the system and x is the input. The transfer function of this system is obtained by taking the Laplace transforms of both sides of Eq. , under the assumption that all initial conditions are zero or

$$\text{Transfer function} = G(s) = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

- The transfer function of a system is a mathematical model in that it is an operational method of expressing the differential equation that relates the output variable to the input variable.
- The transfer function is a property of a system itself, independent of the magnitude and nature of the input or deriving function.
- T.F contains information concerning dynamic behavior, but it does not provide any information concerning the physical structure of the system.
- If T.F of the system is known , the output or response can be studied for various forms of input with a view toward understanding the nature of the system.
- If the T.F of the system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system.



Closed-loop system subjected to a disturbance

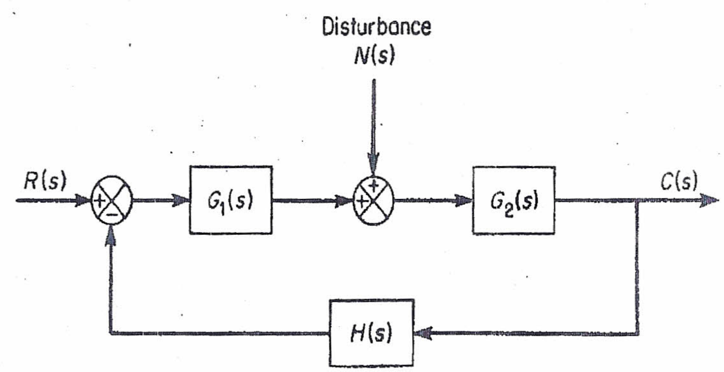
- When two inputs (the reference input and disturbance) are presented in linear closed-loop system , each input can be treated independently of the other.
- In examining the effect of the disturbance N(s) , the input reference R(s) is assumed zero then the response due to disturbance C_N (s) can be found from
- On the other hand , the response due to reference input R(s) can be found assumed zero disturbance input such as
- The total response C(s) then is obtained by adding the two individual responses.

$$\frac{C_N(s)}{N(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

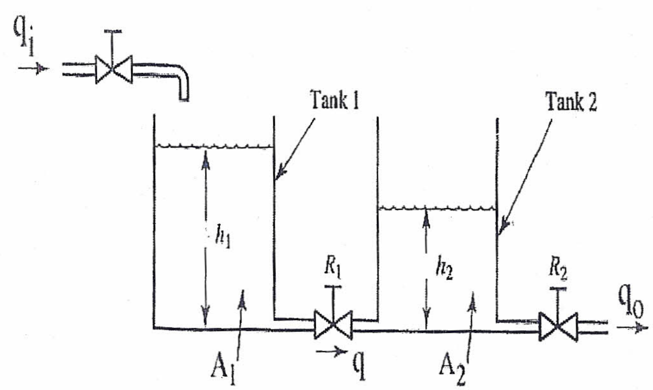
$$C(s) = C_R(s) + C_N(s)$$

$$= \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}[G_1(s)R(s) + N(s)]$$



Example

- For the level tank control find the over all transfer function.

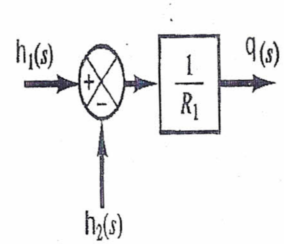


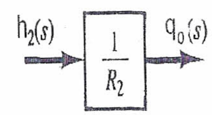
$$\frac{h_1 - h_2}{R_1} = q$$

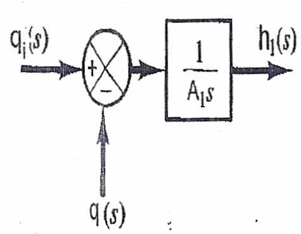
$$A1 \frac{dh_1}{dt} = q_i - q$$

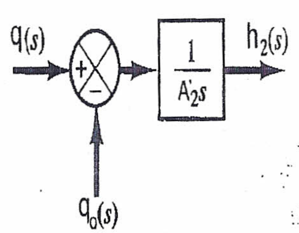
$$\frac{h_2}{R_2} = q_o$$

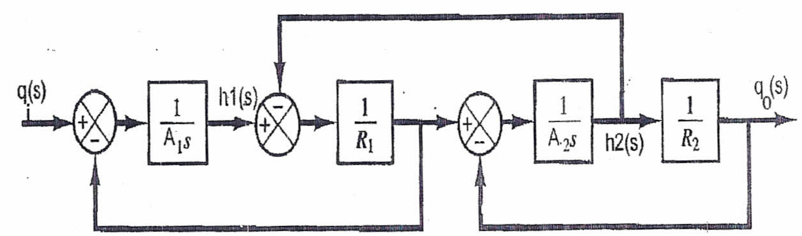
$$A2 \frac{dh_2}{dt} = q - q_o$$

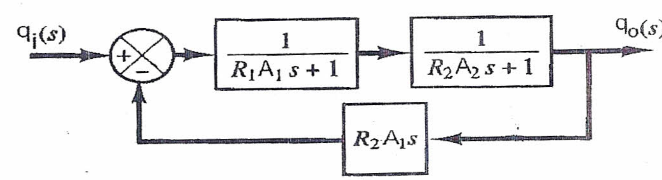


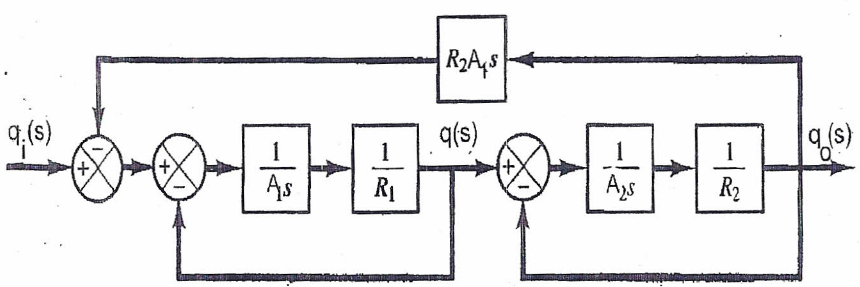


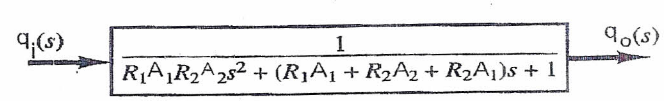






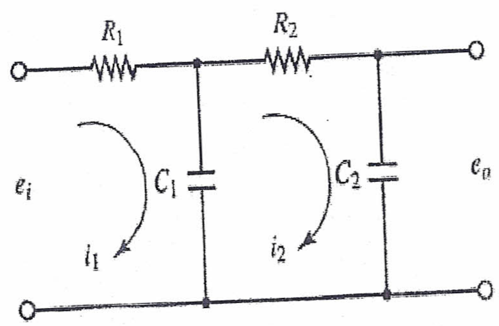






Problems

1. Find the over all transfer function of the Dc motor armature control .
2. For the following system show that the over all T.F is given by



$$E_i(s) \longrightarrow \boxed{\dfrac{1}{R_1C_1R_2C_2s^2+(R_1C_1+R_2C_2+R_1C_2)s+1}} \longrightarrow E_0(s)$$

2-State-space method

- State is the smallest set of variable(called state variable). Or is defined as minimum information required at time to, to find the response to subsequent inputs(t > to)
- State vector is a vector that determines the system state X(t) for any time t ≥ to.

$$x(t) = [x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots, x_n]^T$$

- State space. The n-dimensional space whose coordinate axes consist of the x1 axis, x2 axis,…………, xn axis is called a state space.
- State-space equations

$$\begin{aligned} \dot{x}(t) &= Ax + Bu \\ y(t) &= Cx + Du \end{aligned}$$

- Where A is called the state matrix,
- B is the input matrix
- C is the output matrix
- D is the direct transmission matrix

State-space representation

1- If the forcing function does not involve derivative terms

$$y^n + a_1 y^{n-1} + \dots\dots\dots\dots + a_{n-1}\dot{y} + a_n y = u$$

let

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y} \\ &\vdots \\ x_n &= y^{n-1} \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_n x_1 - a_{n-1}x_2 \dots\dots\dots\dots - a_2 x_{n-1} - a_1 x_n + u \end{aligned}$$

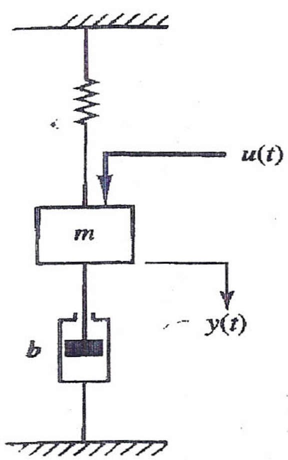
or $\dot{x} = Ax + Bu$

$$y = Cx$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdot & \cdot & \cdot & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

Examples



Consider the mechanical system shown in Figure . We assume that the system is linear. The external force $u(t)$ is the input to the system, and the displacement $y(t)$ of the mass is the output. The displacement $y(t)$ is measured from the equilibrium position in the absence of the external force. This system is a single-input–single-output system.

From the diagram, the system equation is

$$m\ddot{y} + b\dot{y} + ky = u$$

This system is of second order. This means that the system involves two integrators. Let us define state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

Then we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

or

$$\dot{x}_1 = x_2 \qquad (1)$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u \qquad (2)$$

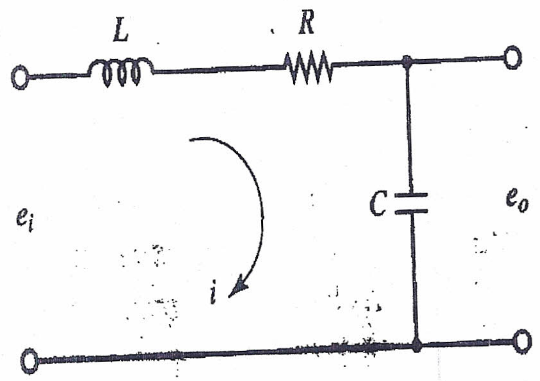
The output equation is

$$y = x_1$$

In a vector-matrix form, Equations (1) and (2) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

Example 2



The sytem model can be written as

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Then by defining state variables by

$$x_1 = e_o$$
$$x_2 = \dot{e}_o$$

and the input and output variables by

$$u = e_i$$
$$y = e_o = x_1$$

we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

These two equations give a mathematical model of the system in state space.

2- If the forcing function involves derivative terms

$$\overset{(n)}{y} + a_1\overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0\overset{(n)}{u} + b_1\overset{(n-1)}{u} + \ldots + b_{n-1}\dot{u} + b_n u$$

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0\dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

$$\vdots$$

$$x_n = \overset{(n-1)}{y} - \beta_0\overset{(n-1)}{u} - \beta_1\overset{(n-2)}{u} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1}u = \dot{x}_{n-1} - \beta_{n-1}u$$

where $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ are determined from

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1\beta_0$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0$$

$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$$

$$\vdots$$

$$\beta_n = b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_n\beta_0$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\vdots$$

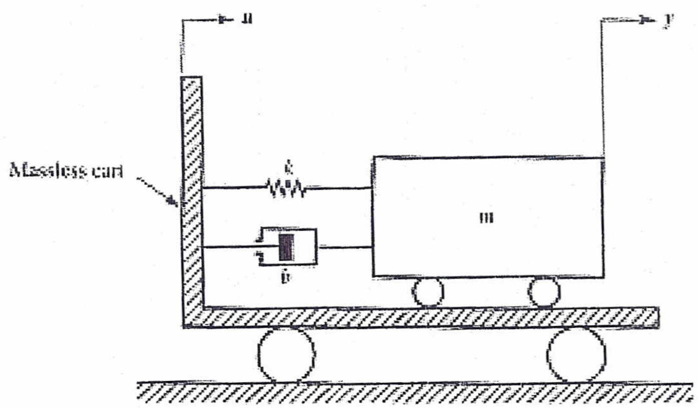
$$\dot{x}_n = -a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n + b_0\overset{(n)}{u} + b_1\overset{(n-1)}{u} + \cdots + b_n u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \beta_0 u$$

Examples

1-For the following system derive the state-system model



$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u)$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

with the standard form

$$\ddot{y} + a_1\dot{y} + a_2y = b_0\ddot{u} + b_1\dot{u} + b_2u$$

and identify $a_1, a_2, b_0, b_1,$ and b_2 as follows:

$$a_1 = \frac{b}{m}, \quad a_2 = \frac{k}{m}, \quad b_0 = 0, \quad b_1 = \frac{b}{m}, \quad b_2 = \frac{k}{m}$$

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = \frac{b}{m}$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2$$

Then

$$x_1 = y - \beta_0 u = y$$

$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u$$

$$\dot{x}_1 = x_2 + \beta_1 u = x_2 + \frac{b}{m}u$$

$$\dot{x}_2 = -a_2x_1 - a_1x_2 + \beta_2 u = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]u$$

and the output equation becomes

$$y = x_1$$

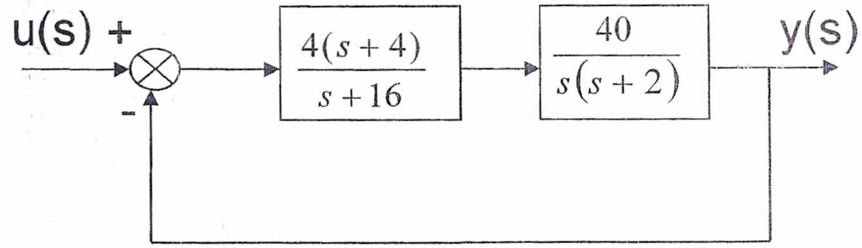
or

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{b}{m}\\ \frac{k}{m} - \left(\frac{b}{m}\right)^2\end{bmatrix}u$$

and

$$y = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

2- Find the state-space for the following system



The corresponding transfer function is

$$\frac{Y(s)}{U(s)} = \frac{160\,(s+4)}{s^3 + 18s^2 + 192\,s + 640}$$

Then, the corresponding differential equation is

$$\dddot{y} + 18\,\ddot{y} + 192\,\dot{y} + 640\,y = 160\,\dot{u} + 640\,u$$

The state-space can be obtained as follows

Let

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$
$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

Where β_0, β_1 and β_2 are determined from

$$\beta_0 = b_0 = 0$$
$$\beta_1 = b_1 - a_1\beta_0 = 0$$
$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = 160$$
$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 = -2240$$

Then the state equation is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_3 & -a_2 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{bmatrix}[u]$$

$$y = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \beta_0 u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -640 & -192 & -18\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 160\\ -2240\end{bmatrix}[u]$$

$$y = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Correlation between Transfer-function and State-space

- Let the state equation is given by

$$\dot{x}(t) = Ax + Bu$$
$$y(t) = Cx + Du$$

- Using Laplace transform technique yields

$$sX(s) = AX(s) + BU(s)$$
$$(sI - A)X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1}BU(s)$$
$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$
$$G(s) = \frac{Y(s)}{U(s)} = [C(sI - A)^{-1}B + D] = C\phi(s)B + D$$
$$\phi(s) = (SI - A)^{-1} \text{ and}$$
$$\text{this is called State - transaition matrix}$$

Example :Find the Transfer function for the following system

$$\dot{x} = \begin{bmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u \quad \text{and} \quad y = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then $$[SI - A]^{-1} = \frac{1}{S^2 + \frac{R}{L}S + \frac{1}{LC}}$$

$$G(s) = C[SI - A]^{-1} B = \frac{R/LC}{S^2 + \frac{R}{L}S + \frac{1}{LC}}$$

Solving the time-invariant state equation

1. Homogeneous sate equation

- **Scalar differential equation** $\dot{x} = ax$ **(1)**

The solution can be assumed as polynomial $x(t) = b_0 + b_1 t + b_2 t^2 + \dots\dots\dots + b_k t^k + \dots$

By substituting this into equation 1 yields

$$b_1 + 2b_2 t + 3b_3 t^2 \dots\dots\dots + kb_k t^{k-1} + \dots = a\left(b_0 + b_1 t + b_2 t^2 + \dots\dots\dots + b_k t^k + \dots\right) \qquad (2)$$

Equating the coefficients of the equal powers of t yields

$$b_1 = ab_0$$

$$b_2 = \frac{1}{2} ab_1 = \frac{1}{2} a^2 b_0$$

$$b_3 = \frac{1}{3} ab_2 = \frac{1}{3 * 2} a^3 b_0$$

$$\vdots$$

$$b_k = \frac{1}{k!} a^k b_0$$

Where b_0 can be determined by substituting t=0 into equation 2

$x(0) = b_0$

Then, the solution is

$$x(t) = \left(1 + at + \frac{1}{2!} a^2 t^2 + \dots\dots + \frac{1}{k!} a^k t^k + \dots.\right) x(0)$$

$$= e^{at} x(0)$$

- **Vector matrix differential equation** $$\dot{x} = Ax$$

Where x = n-vector A = n*n constant matrix

If x is defined as

$$x(t) = b_0 + b_1 t + b_2 t^2 + \ldots\ldots\ldots + b_k t^k + \ldots$$

$$b_1 = A b_0$$

$$b_2 = \frac{1}{2} A b_1 = \frac{1}{2} A^2 b_0$$

$$b_3 = \frac{1}{3} A b_2 = \frac{1}{3 * 2} A^3 b_0$$

.

.

$$b_k = \frac{1}{k!} A^3 b_0$$

The solution is

$$x(t) = \left(I + At + \frac{1}{2!} A^2 t^2 + \ldots\ldots + \frac{1}{k!} A^k t^k + \ldots \right) x(0)$$

$$= e^{At} x(0) = \phi(t) x(0)$$

Where φ(t) is called State-transition matrix

Laplace transform approach to find the solution of homogeneous state equations

- Let $\dot{x} = Ax$
- Taking the Laplace transform of both sides yields $sx(s) - x(0) = Ax(s)$
- Or

$$(sI - A)x(s) = x(0)$$

- Premultiplying both sides by (SI – A) $^{-1}$ yields X(s) = (sI – A) $^{-1}$ x(0)

- Where $(sI - A)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \ldots.$

- Then , take the inverse Laplace transform yields

$$\Im^{-1}[(sI-A)^{-1}] = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots. = e^{At}$$

- Therefore , the solution is

$$x(t) = e^{At}x(0) = \phi(t)x(0)$$

Properties of state-transition matrices

- For the time-invariant system

1- $$\phi(0) = e^{A(0)} = I$$

2- $$\phi(t) = e^{At} = \left(e^{-At}\right)^{-1} = [\phi(-t)]^{-1}$$

3- $$\phi^{-1}(t) = \phi(-t)$$

4- $$\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1}e^{At_2} = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$$

5- $$\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0) = \phi(t_1 - t_0)\phi(t_2 - t_1)$$

6- $$[\phi(t)]^n = \phi(nt)$$

Examples

1. Obtain the state-transition **φ(t)** and inverse state-transition matrix **φ-1(t)** of the following

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For this system

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Then

$$\varphi(t) = e^{At} = \mathfrak{I}^{-1}\left[(sI - A)^{-1}\right] \qquad sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\phi^{-1}(t) = e^{-At} = \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

problem

- Find the solution of the control systems described by

1-
$$\ddot{y}+5\dot{y}+6y=f(t)$$
$$where \quad f(t)=0, \quad y(0)=1, \quad \dot{y}(0)=0$$

2-
$$\ddot{y}+10\dot{y}+25y=f(t)$$
$$where \quad f(t)=0, \quad y(0)=1, \quad \dot{y}(0)=0$$

2- **non-Homogeneous sate equation**

$$\dot{x}(t) = Ax + Bu$$
$$y(t) = Cx + Du$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \varphi(t)x(0) + \int_0^t \varphi(t-\tau)Bu(\tau)d\tau$$

Example : obtain the time response of the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Where u(t) is the unit step function applied at t = 0. u(t) = 1

$$\phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\phi(t-\tau) = \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix}$$

The response to the unit-step input is then obtained as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\, d\tau$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

In the special case where the initial state is zero, or $\mathbf{x}(0) = \mathbf{0}$, the solution $\mathbf{x}(t)$ can be simplified to

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Sylvester's interpolation formula for compute (e^{At})

- The solution of e^{At} can be obtained by solving the following determinant equation

$$\begin{vmatrix} 1 & \lambda_1 & \lambda^2{}_1 & . & . & . & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_2 & \lambda^2{}_2 & & & & \lambda_2^{m-1} & e^{\lambda_1 t} \\ . & . & . & & & & . & . \\ . & . & . & & & & . & . \\ . & . & . & & & & . & . \\ 1 & \lambda_m & \lambda^2{}_m & . & . & . & \lambda_m^{m-1} & e^{\lambda_m t} \\ I & A & A^2 & . & . & . & A^{m-1} & e^{At} \end{vmatrix} = 0$$

- By solving the above determinant e^{At} can be obtained in terms of the A^k (k = 0, 1,2, m-1)
- Or it can be solved by writing the solution as

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2 + \ldots. + \alpha_{m-1}(t)A^{m-1}$$

Example

- Compute e^{At} using Sylvester's interpolation formula if $A=\begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}$

$$\begin{vmatrix}1 & \lambda_1 & e^{\lambda_1 t}\\ 1 & \lambda_2 & e^{\lambda_2 t}\\ I & A & e^{At}\end{vmatrix}=0$$

- Where λ are the eigenvalues of matrix A, and are the roots of the characteristic equation

$$|\lambda I - A| = 0$$

- Then

$$\left|\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}-\begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}\right|=0$$

$$\begin{vmatrix}\lambda & -1\\ 0 & \lambda+2\end{vmatrix}=0$$

$$\lambda(\lambda+2)=0 \qquad \lambda_1=0, \quad \lambda_2=\text{-}2$$

$$\begin{vmatrix}1 & 0 & 1\\ 1 & -2 & e^{-2t}\\ I & A & e^{At}\end{vmatrix}=0$$

$$or \quad -2e^{At}-Ae^{-2t}+A+2I=0$$

$$or \quad e^{At}=\frac{1}{2}(A+2I-Ae^{-2t})$$

$$=\begin{bmatrix}1 & \frac{1}{2}\left(1-e^{-2t}\right)\\ 0 & e^{-2t}\end{bmatrix}$$

- An alternative approach is

$$\alpha_0(t)+\alpha_1(t)\lambda_1 = e^{\lambda_1 t}$$
$$\alpha_0(t)+\alpha_1(t)\lambda_2 = e^{\lambda_2 t}$$
$$\lambda_1 = 0, \qquad \lambda_2 = -2$$
$$\alpha_0(t) = 1$$
$$\alpha_0(t) - 2\alpha_1(t) = e^{-2t}$$
$$\alpha_1(t) = \frac{1}{2}(1-e^{-2t})$$

- Then

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A = I + \frac{1}{2}\left(1-e^{-2t}\right)A$$
$$= \begin{bmatrix} 1 & \frac{1}{2}\left(1-e^{-2t}\right) \\ 0 & e^{-2t} \end{bmatrix}$$

- Problem compute e^{At} by two methods if $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

- Example

1. Find the equilibrium points for the system described by

$$\ddot{y} + (1 + y)\dot{y} - 2y + 0.5y^3 = 0$$

2. Then evaluate the linearized Jacobian matrix at each equilibrium points.

3. Determine the stability characteristics for the eigenvalues.

Solution

- Find the state variable model

Let $x_1 = y \ and \ x_2 = \dot{y}$ gives the state variable model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 2x_1 - 0.5x_1^3 - (1 + x_1)x_2 \end{bmatrix} = f(x)$$

- Equilibrium points are solutions of f(x) = 0, so each must have

$$x_2 = 0 \quad \text{and} \quad 2x_1 - 0.5x_1^3 = 0.$$

- The three solutions are $x_{e1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x_{e2} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{ and } x_{e3} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$

- The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 2 - \dfrac{3x_1^2}{2} - x_2 & -(1 + x_1) \end{bmatrix} \quad \text{so that}$$

- $\left[\dfrac{\partial f}{\partial x}\right]_1 = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ its eigenvalues are +1 and -2

- $\left[\dfrac{\partial f}{\partial x}\right]_2 = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix}$ its eigenvalues are $-\dfrac{3}{2} \pm j\sqrt{7/2}$

$\left[\dfrac{\partial f}{\partial x}\right]_3 = \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix}$ its eigenvalues are $\dfrac{1}{2} \pm j\sqrt{15/2}$

Matlab software

1. To represent a polynomial in Matlab

$$p(s) = s^3 + 3s^2 + 4$$

2. Define the polynomial as
3. p=[coefficients of the polynomial]
4. p=[1 3 0 4];
5. To find the roots of the polynomial use
6. R = roots(p)

7. When the product of two polynomial is required the command (CONV) is used
   - Find the product of

$$p = 3s^2 + 2s + 1$$

$$q = s + 4$$

   - p=[3 2 1], q =[1 4]
   - M=conv (p,q)

How to represent the Transfer function in Matlab

- To represent the T.F in Matlab the following steps can be followed
- Define the numerator polynomial as
- Num=[coefficients of numerator polynomial];
- Define the denominator polynomial as
- Den=[coefficients of denominator polynomial];
- Sys = tf (num, den)

$$G(s) = \frac{10s^2 + 5s + 1}{s^3 + 4s^2 + 2s + 1}$$

- Num=[10 5 1];
- Den=[1 4 2 4];
- Sys = tf(num,den)
- P=pole(sys)
- Z=zero(sys)

Transformation from T.F to state-space or vice-versa

- The command to perform the conversion is
- [A,B,C,D]=tf2ss(num,den);
- Printsys(A,B,C,D)
- Or
- [num,den]=ss2tf(A,B,C,D,iu)
- % where iu is specified for more than one input.
- Sys=tf(num,den)
- Let

$$G(s) = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}$$

- Num=[2 8 6]; den=[1 8 16 6];

$$A = \begin{bmatrix} -8 & -16 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 8 & 6 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

- A = [-1 -1;6.5 0]
- B = [1; 1]
- C =[1 0]
- D = [1]

$$G(s) = \frac{s^2 + 2s + 5.5}{s^2 + s + 6.5}$$

- For a system with multiple inputs and multiple outputs;
- Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

- A=[0 1; -25 -4]; B=[1 1; 0 1]; C=[1 0 ; 0 1]; D=[0 0 ; 0 0];
- [n1, d1]=ss2tf(A,B,C,D,1) (to find transfer functions due to input 1)
- [n2,d2]=ss2tf(A,B,C,D,2) (to find transfer functions due to input 2)

```
n1 = 0   1    4          d1 = 1  4  25
     0   0  -25

n2 = 0   1    5          d2 = 1   4   25
     0   1  -25
```

- n11=[n1(1,1) n1(1,2) n1(1,3)]
- n21=[n2(1,1) n2(1,2) n2(1,3)]

```
n11 = 0   1    4
n12 = 0   0  -25
```

- n12=[n1(2,1) n1(2,2) n1(2,3)]
- n22=[n2(2,1) n2(2,2) n2(2,3)

```
n21 = 0 1  5
n22 = 0 1 -25
```

- Sys11=tf(n11,d1,1)

$$\frac{Y_1(s)}{U_1(s)} = \frac{s+4}{s^2+4s+25},$$

- Sys21=tf(n21,d2,1)

$$\frac{Y_2(s)}{U_1(s)} = \frac{-25}{s^2+4s+25}$$

- Sys12=tf(n12,d1,2)

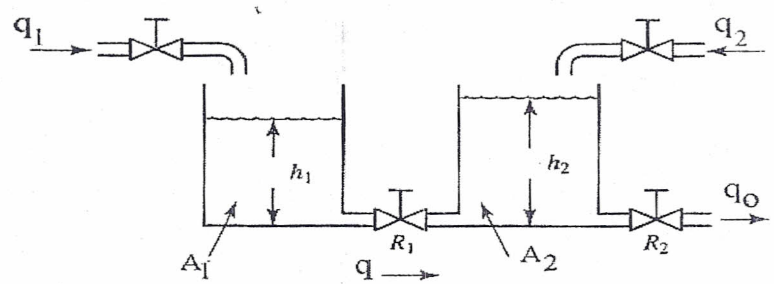
$$\frac{Y_1(s)}{U_2(s)} = \frac{s+5}{s^2+4s+25},$$

- Sys22=tf(n22,d2,2)

$$\frac{Y_2(s)}{U_2(s)} = \frac{s-25}{s^2+4s+25}$$

Problems

- Find the state space model to level-control system



- Find the Transfer function of the following system

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

State-space representation forms

- The state space can be represented by four forms
- Let a system defined by

$$y^{n} + a_1 y^{n-1} + a_2 y^{n-2} + \ldots\ldots\ldots + a_{n-1}\dot{y} + a_n y = b_o u^{n} + b_1 u^{n-1} + \ldots\ldots + b_{n-1}\dot{u} + b_n u$$

1. Controllable canonical form

This form is important in control system design. This form is

A state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdot & \cdot & \cdot & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

Output state equation is

$$y = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdot & \cdot & \cdot & \cdot & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

- ## Observable canonical form

This form is important for design an observer estimator . This has the following form

A state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & . & . & 0 & -a_n \\ 1 & 0 & 0 & . & . & 0 & -a_{n-1} \\ 0 & 1 & 0 & . & . & . & -a_{n-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ . \\ . \\ . \\ . \\ b_1 - a_1 b_0 \end{bmatrix} u$$

Output state equation is

$$y = \begin{bmatrix} 0 & 0 & . & . & . & . & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

3- Diagonal canonical form

This can be derived only when the denominator polynomial involved only distinct real roots.

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots\dots + b_{n-1} s + b_n}{(s+p_1)(s+p_2)\dots\dots\dots(s+p_n)}$$

$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \dots\dots + \frac{c_n}{s+p_n}$$

A state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -p_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix} u$$

Output state equation is

$$y = \begin{bmatrix} c_1 & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

4- Jordan canonical form

This form is used when there are multiple roots in the denominator.

let $p_1 = p_2 = p_3$ or

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots\ldots + b_{n-1}s + b_n}{(s+p_1)^3(s+p_4)\ldots\ldots\ldots(s+p_n)}$$

$$= b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{(s+p_1)} + \frac{c_4}{s+p_4} + \ldots\ldots + \frac{c_n}{s+p_n}$$

A state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & -p_1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & -p_1 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & -p_4 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \cdot \\ 1 \\ 1 \end{bmatrix} u$$

Output state equation is

$$y = \begin{bmatrix} c_1 & c_2 & \cdot & \cdot & \cdot & \cdot & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Example

- Obtain state-space representations in the controllable canonical form, observable canonical form, and diagonal canonical form if the transfer function of the system is

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2} \text{ or } \ddot{y}+3\dot{y}+2y=\dot{u}+3u$$

$$\ddot{y}+a_1\dot{y}+a_2y=b_o\ddot{u}+b_1\dot{u}+b_2u$$

- n=2, $a_2 = a_n = 2$, $a_1=3$ $b_o=0$, $b_1=1$ and $b_2=3$

1. Controllable canonical form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a_2 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y(t) = \begin{bmatrix}b_2-a_2b_0 & b_1-a_1b_0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + b_0u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y(t) = \begin{bmatrix}3 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

2. observable canonical form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & -a_2\\ 1 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}b_2-a_2b_0\\ b_1-a_1b_0\end{bmatrix}u$$

$$y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+b_0u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}3\\ 1\end{bmatrix}u$$

$$y(t)=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

3. Diagonal canonical form

$$\frac{Y(s)}{U(s)}=\frac{b_0s^n+b_1s^{n-1}+\dots\dots+b_{n-1}s+b_n}{(s+p_1)(s+p_2)\dots\dots\dots(s+p_n)}$$

$$=b_0+\frac{c_1}{s+p_1}+\frac{c_2}{s+p_2}+\dots\dots+\frac{c_n}{s+p_n}$$

$$\frac{Y(s)}{U(s)}=\frac{s+3}{s^2+3s+2}=\frac{b_0s^2+b_1+b_2}{(s+p_1)(s+p_2)}=\frac{s+3}{(s+1)(s+2)}$$

$$\frac{Y(s)}{U(s)}=b_0+\frac{c_1}{s+p_1}+\frac{c_2}{s+p_2}=\frac{2}{s+1}+\frac{-1}{s+2}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-p_1 & 0\\ 0 & -p_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y(t)=\begin{bmatrix}c_1 & c_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+b_0u$$

$$y(t)=\begin{bmatrix}2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

problems

1- Obtain the state-space representation for the following system in

(a) controllable canonical form .

(b) observable canonical form

$$\frac{Y(s)}{U(s)} = \frac{s+6}{s^2+5s+6}$$

2- Obtain a state-space in a diagonal canonical form

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$$

3- Find x1(t) and x2(t) of the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Where the initial conditions are

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Linearized using state variable

- Consider the following general nonlinear state variable model

$$\dot{x} = f(x,u,t)$$

$$y = h(x,u,t) \qquad \text{eq(1)}$$

- Assume a nominal solution $X_n(t)$, $u_n(t)$, and $y_n(t)$ is known . The difference between these nominal vector functions and some slightly functions x(t), u(t), and y(t) can be defined by

$$\delta x = x(t) - x_n(t)$$

$$\delta u = u(t) - u_n(t)$$

$$\delta y = y(t) - y_n(t)$$

$$\dot{x}_n + \delta\dot{x} = f(x_n + \delta x, u_n + \delta u, t)$$

Then

$$= f(x_n, u_n, t) + \left[\frac{\partial f}{\partial x}\right]_n \delta x + \left[\frac{\partial f}{\partial u}\right]_n \delta u + \text{higher order term}$$

$$\mathrm{y}_n + \delta y = h(x_n + \delta x, u_n + \delta u, t)$$

$$= h(x_n, u_n, t) + \left[\frac{\partial h}{\partial x}\right]_n \delta x + \left[\frac{\partial h}{\partial u}\right]_n \delta u + \text{higher order term}$$

- Where [$]_n$ means the derivative are evaluated on nominal solutions. Since the nominal solutions satisfy eq (1) , the first terms in Taylor series expansions cancel.

- For sufficiency small δx , δu and δy , the higher –order terms can be neglected, Then

$$\delta\dot{x} = \left[\frac{\partial f}{\partial x}\right]_n \delta x + \left[\frac{\partial f}{\partial u}\right]_n \delta u$$

$$\delta y = \left[\frac{\partial h}{\partial x}\right]_n \delta x + \left[\frac{\partial h}{\partial u}\right]_n \delta u$$

- If $x_n(t) = x_e$ = constant and if $u_n(t) = 0 = \delta_u(t)$, then the stability of the equilibrium point x_e is governed by

$$\delta\dot{x} = \left[\frac{\partial f}{\partial x}\right]_n \delta x$$

- For this case , the system stability can be determine as following

1. If all eigenvalues of the Jacobian matrix ([$\delta f/ \delta x$]) have negative real parts, the equilibrium point (f(x)=0) is asymptotically stable for sufficiently small perturbation otherwise it will be unstable

2. If one or more of the eigenvalues are on the jω axis and all others in the left-half plane, no conclusion about stability can be drawn from this model.

3. Whether the actual behavior of the system is divergent or convergent will depend upon the neglected higher – order terms in the Taylor expansive .

4. Except for the borderline jω axis case, stability of the nonlinear is the same as the linearized model at least near the equilibrium point .

5. Equilibrium point is a point where the system can stay forever without moving

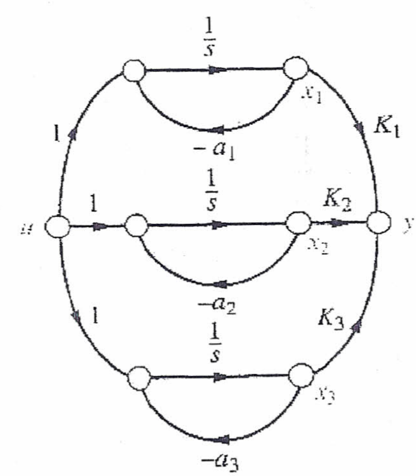
6. Nonlinear systems frequently have more than one equilibrium point.

Controllability

- Controllability is the one of the important objective of the modern control .
- A system is said to be completely state controllable if

For any initial time to , each initial state X(to) can be transferred to any final state X (tf) in a finite time tf > to by means of an unconstrained control U(t).

- Unconstrained control U(t) has no limit on the amplitudes.
- Consider the following system . It is clear that all initial state variable $X_n(t)$ can be controlled by the input signal u(t)

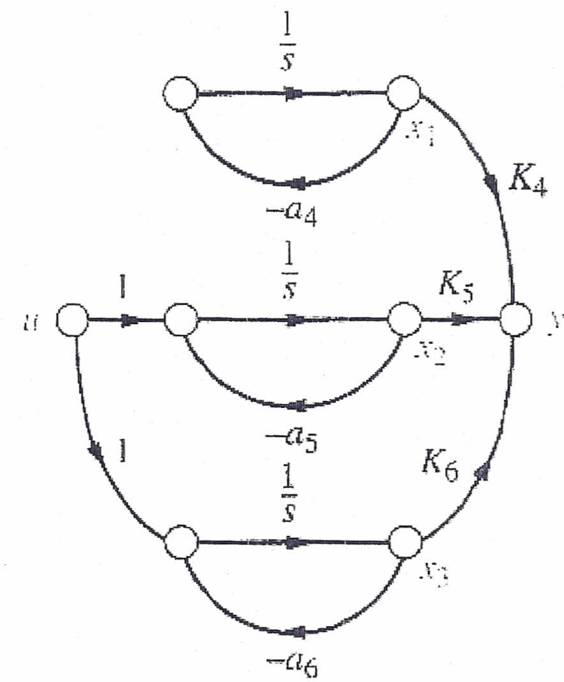


$$\dot{x}_1 = -a_1 x_1 + 0x_2 + 0x_3 + u$$
$$\dot{x}_2 = 0x_1 - a_2 x_2 + 0x_3 + u$$
$$\dot{x}_3 = 0x_1 + 0x_2 - a_3 x_3 + u$$
$$y = k_1 x_1 + k_2 x_2 + k_3 x_3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- But for the system given below, the state variable X1 is not controlled by control u(t) . Then the system is uncontrollable .



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_4 & 0 & 0 \\ 0 & -a_5 & 0 \\ 0 & 0 & -a_6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} k_4 & k_5 & k_6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- **Example 2**

- Determine whether the following system is controllable or not

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

- controllability matrix is

$$M_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$$

- Rank of Mc =1 $\neq$ n
- determinant =0
- Then the above system does not satisfied the controllability condition and it is

uncontrollable.

- ## Method II
- If the system is represented by canonical controllable form with non-repeated eigenvalues , then the controllability condition can be derived in the following

1. Transfer the canonical controllable form into diagonal form by using the transformation

$$x = Tz$$

1. Then check the new matrix **B'**.
2. If matrix **B'** has no zero row , then the system is controllable
3. This can be done by the following

$$\dot{x} = Ax + Bu$$

$$T\dot{z} = ATz + Bu$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu = \Lambda z + B'u$$

$$where \quad \Lambda = T^{-1}AT, \quad B' = T^{-1}B$$

4. Where transformation matrix T can be found from

$$T = \begin{bmatrix} 1 & 1 & . & . & . & 1 \\ \lambda_1 & \lambda_2 & . & . & . & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & . & . & . & \lambda_n^2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \lambda_1^{n-1} & \lambda_2^{n-1} & . & . & . & \lambda_n^{n-1} \end{bmatrix}$$

- Example use the transformation method state whether the following system is controllable or not.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

- The eigenvalues can be found by $|\lambda I - A| = 0$,$\lambda 1$ = j and $\lambda 2$ = -j
- Then

$$T = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$$

$$T^{-1} = \frac{1}{-2j} \begin{bmatrix} -j & -1 \\ -j & 1 \end{bmatrix}$$

- Therefore

$$B' = T^{-1} B = \frac{1}{-2j} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- The B′ has no zero row , then the system is completely state controllable

Condition for complete state controllability in the s-plane

- A necessary and sufficient condition for complete state controllability is that no cancellation occur in the transfer function.
- Ex consider the following transfer function $$\frac{Y(s)}{U(s)} = \frac{s+2.5}{(s+2.5)(s-1)}$$
- Clearly, cancellation of factor (s + 2.5) is occurred , then the system is uncontrollable.
- The state space form is $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
- controllability matrix is $$M_c = [B \quad AB] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
- Rank of Mc =1 $\neq$ n
- determinant =0
- Then the above system does not satisfied the controllability condition and it is uncontrollable.

Output controllability

- In some cases, it is desired to control the output rather than the state of the system .
- A system is said to be output controllable if it is possible to construct an unconstrained vector U(t) that will transfer any given initial output y (t_0) to any final output y (t_f) in a finite time $t_0 \le t \le t_f$.
- Any system described by state-space is completely output controllable if

$$rank\ \mathrm{M}_o = \begin{bmatrix} CA & CAB & CA^2B & . & . & . & CA^{n-1} & D \end{bmatrix} = m$$
$$where \quad m = no\ \ of\ \ rows\ \ of\, C\ matrix$$

- Where Mo is called output matrix
- Example : consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- The Mo is

$$M_0 = \begin{bmatrix} CB & CAB & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

- The rank of Mo = 1= m
- Then the system is completely output controllable

How to obtain a controllable canonical form

Steps to obtain the controllable canonical form

1. The system must be completely state controllable.

2. Find a characteristic equation $|SI - A| = s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1}s + a_n$

3. Define a transformation matrix T as $T = M_c * W$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & . & . & . & a_1 & 1 \\ a_{n-2} & a_{n-3} & . & . & . & 1 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1 & 1 & . & . & . & 0 & 0 \\ 1 & 0 & . & . & . & 0 & 0 \end{bmatrix}$$

Where a_i's are the coefficients of the characteristic equation

4- Define a new state vector $\widetilde{x}$

$$x = T\widetilde{x}$$

$$\dot{x} = Ax + Bu$$

$$T\dot{\widetilde{x}} = AT\widetilde{x} + Bu$$

$$\dot{\widetilde{x}} = T^{-1}AT\widetilde{x} + T^{-1}Bu = \widetilde{A}\widetilde{x} + \widetilde{B}u$$

where $$\widetilde{A} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & \\ & & & & & & \\ 0 & 0 & & & & & 1 \\ -a_n & -a_{n-1} & & & & & -a1 \end{bmatrix}$$

$$\widetilde{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

EXAMPLE

- Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Transfer the system equation into controllable canonical form

1. The controllability matrix is

$$\mathrm{M_c} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 \\ 0 & 0 & 1 \\ 1 & -3 & 9 \end{bmatrix}$$

2. The rank of controllability matrix = 3 = n
3. Thus the system is completely state controllable
4. Characteristic equation is $|SI - A| = s^3 + 6s^2 + 11s + 6$

$$a_n = a_3 = 6 \quad , \quad a_{n-1} = a_2 = 11, \quad a_{n-2} = a_1 = 6$$

- The Transformation matrix T is

$$T = M_c W$$

$$\text{where} \qquad W = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 1 & -4 \\ 0 & 0 & 1 \\ 1 & -3 & 9 \end{bmatrix} \begin{bmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

- The inverse matrix is

$$T^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ -3 & 4 & 1 \end{bmatrix}$$

- Then

$$\tilde{A} = T^{-1} A T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

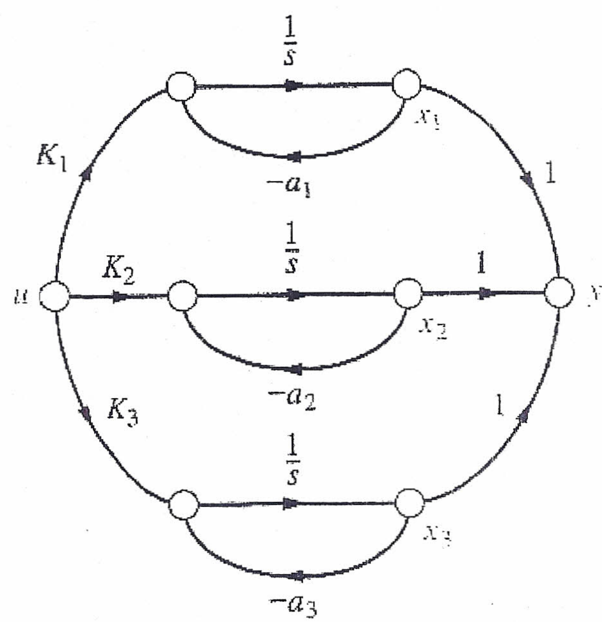
$$\tilde{B} = T^{-1} B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Observability

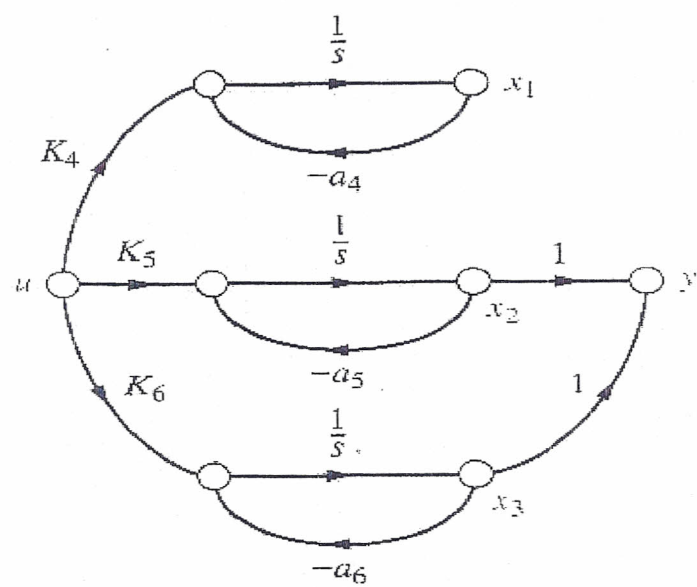
- If the initial state vector , X(t0), can be determined from the observation of y(t) measured over a finite interval of time from t0 [t0 $\leq$ t $\leq$ tf]. the system is said to be completely observable.
- If all state variable has effect on the output , then the system is said to be observable. For example if the output equation for the diagonalized system with distinct eigenvalues is given by

$$Y = Cx = [\,1 \;\; 1 \;\; 1\,]\,x$$

Then the system is observable



- If any state variable has no effect upon the output , then it can not evaluate this state-variable by observing the output . Then the system is called unobservable.
- If the output equation is $Y = [\,0\ 1\ 1\,]\,x$ then the system unobservable



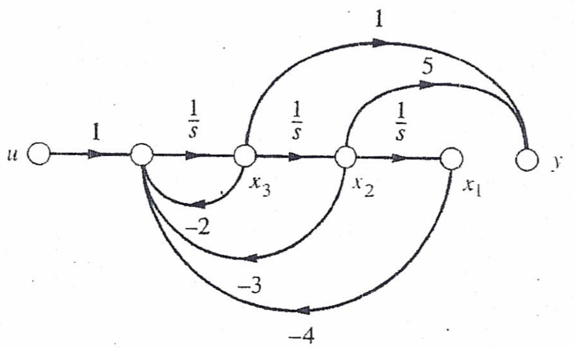
- The concept of Observability is useful in solving the problem of reconstructing unmeasurable state variables from measurable variables in the minimum possible length of time.

Condition for Observability

- The system is completely state observable if the rank of the following[nmxn] matrix is equal n.

$$M_0 = \begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

- The above matrix is called Observability matrix.
- Or the determinant of the matrix $\neq 0$
- **Example 1**
- Consider the system given by



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

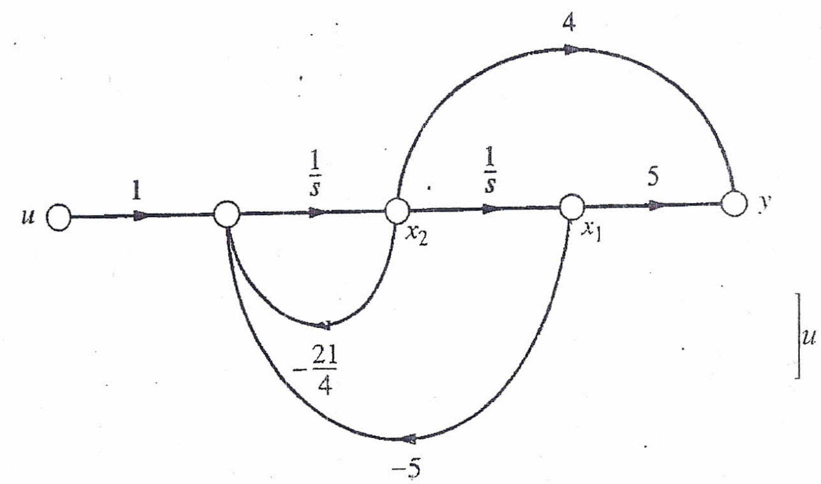
$$y = \begin{bmatrix} 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- State whether the above system is observable or Not
- From inspection ,it can be concluded that the system is unobservable . This conclusion is valid only for diagonalized system with distinct eigenvalues.
- Observability matrix is

$$M_0 = \begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 5 & 1 \\ -4 & -3 & 3 \\ -12 & -13 & -9 \end{bmatrix}$$

- Rank of Mo =3 = n
- Then the above system satisfied the Observability condition and it is Observable.

- Example 2
- Determine if the system given below is observable



- Solution

- The state and output equations for the system are

- The Observability matrix is

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -20 & -16 \end{bmatrix}$$

- The determinant for this Observability matrix equals 0 . Thus, the Observability matrix does not have full rank, and the system is not observable

Condition for complete Observability in the S-plane

- A necessary and sufficient condition for complete state Observability is that no cancellation occur in the transfer function.
- Consider the following system

$$\frac{C(s)}{R(s)} = \frac{s+2}{(s+2)(s+5)}$$

- Show that if the cancellation is occurred the system is unobservable

How to obtain a observable canonical form

Steps

1. The system must be completely state observable.

2. Find a characteristic equation $|SI - A| = s^n + a_1 s^{n-1} + \dots\dots + a_{n-1} s + a_n$

3. Define a transformation matrix Q as $Q = (W * M_o)^{-1}$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & . & . & . & a_1 & 1 \\ a_{n-2} & a_{n-3} & . & . & . & 1 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1 & 1 & . & . & . & 0 & 0 \\ 1 & 0 & . & . & . & 0 & 0 \end{bmatrix}$$

Where a_i's are the coefficients of the characteristic equation

4- Define a new state vector $\widetilde{x}$

$$x = Q\widetilde{x}$$

$$\dot{x} = Ax + Bu$$

$$Q\dot{\widetilde{x}} = AQ\widetilde{x} + Bu$$

$$\dot{\widetilde{x}} = Q^{-1}AQ\widetilde{x} + Q^{-1}Bu = \widetilde{A}\widetilde{x} + \widetilde{B}u$$

where $\widetilde{A} = \begin{bmatrix} 0 & 0 & 0 & . & . & . & -a_n \\ 1 & 0 & 0 & . & . & . & -a_{n-1} \\ . & . & . & . & . & . & \\ & & & & & & \\ 0 & 0 & & & & & 1 \\ 0 & 0 & & & & 1 & -a1 \end{bmatrix}$

$$\widetilde{B} = \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ . \\ \hline . \\ . \\ . \\ b_1 - a_1 b_0 \end{bmatrix} \qquad \widetilde{\mathrm{C}} = \mathrm{CQ} = [0 \quad 0 \quad . \quad 1]$$

Example

- Consider a system defined by $\dot{X} = AX + Bu = \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$

$$y = CX = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

- Observability matrix is $M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}$

- Characteristic equation is $|SI - A| = s^2 + 2s + 1 = s^2 + a_1 s + a_2$
- Then $a_n = a_2 = 1 \quad , \quad a_{n-1} = a_1 = 2$

- The transformation matrix Q is

$$W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

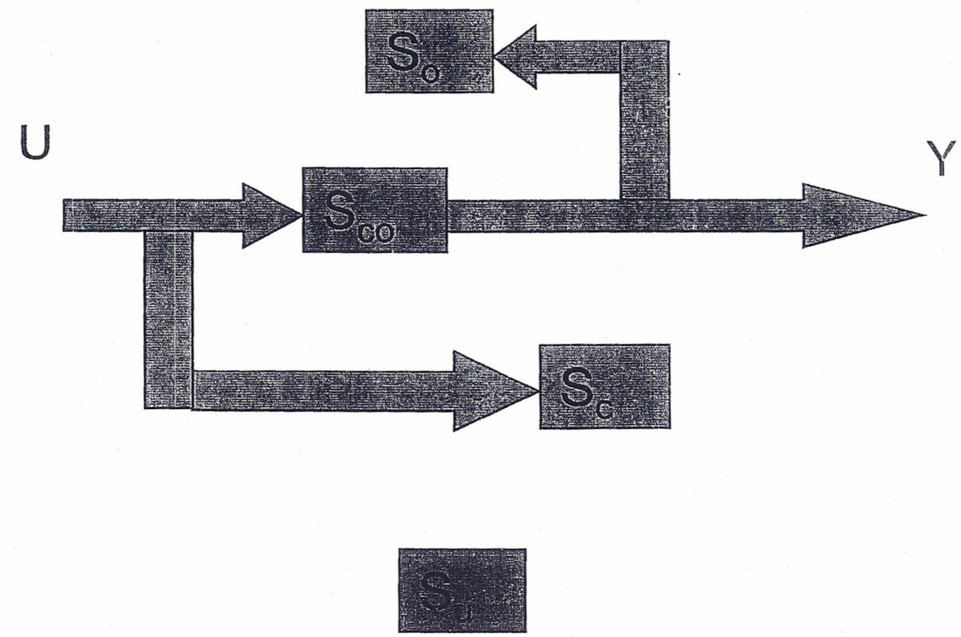
$$Q = (W * M_o)^{-1} = \left\{ \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \right\}^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\tilde{A} = Q^{-1} A Q = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

$$\tilde{B} = Q^{-1} B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \tilde{C} = CQ = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

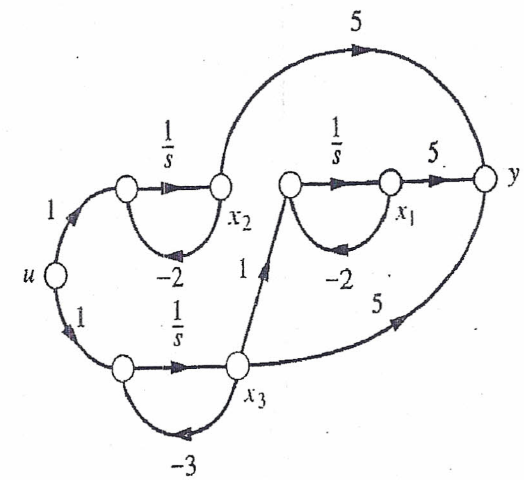
- In general the controllability and Observability can be illustrated graphically by

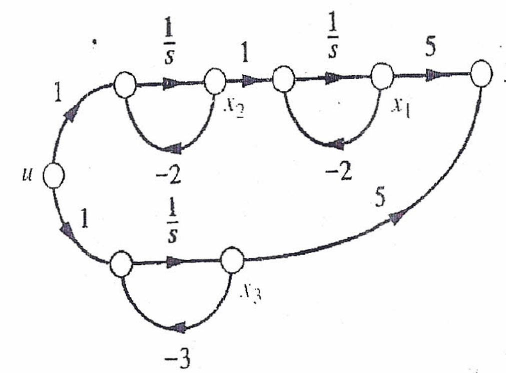


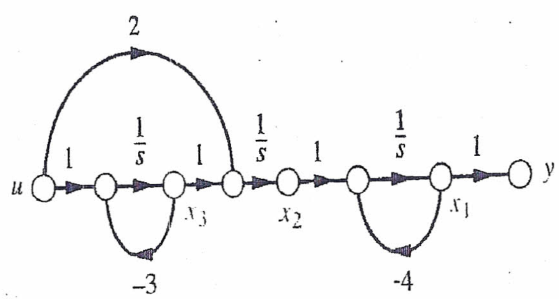
- Sco = completely controllable and completely Observable.
- Sc = Controllable but unobservable.
- So = Observable but uncontrollable.
- Su = Uncontrollable and unobservable

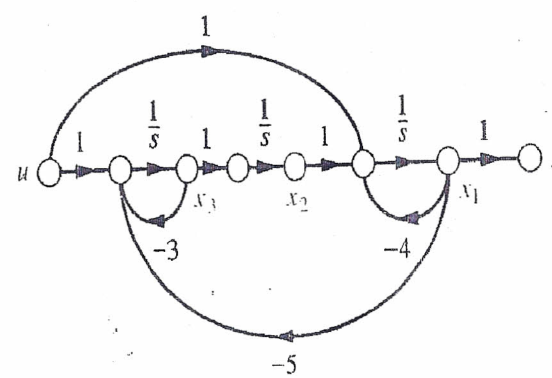
problems

1. For the following plants , determine the controllability . If the controllability can be determined by inspection, state that it can and then verify your conclusions using the controllability matrix







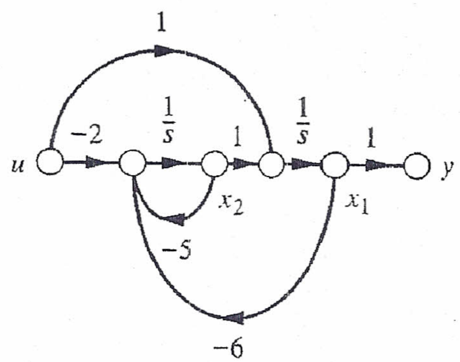


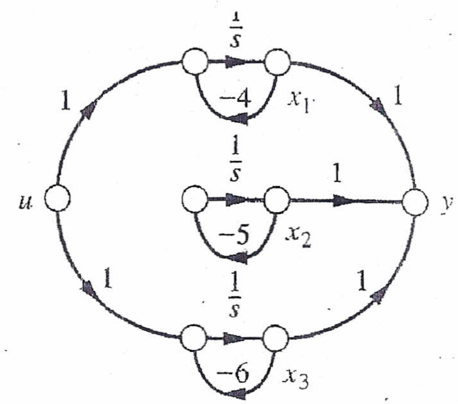
2. Determine whether the system given below is Observable or not.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -3 \\ 0 & -2 & 1 \\ -7 & -8 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} [u]$$

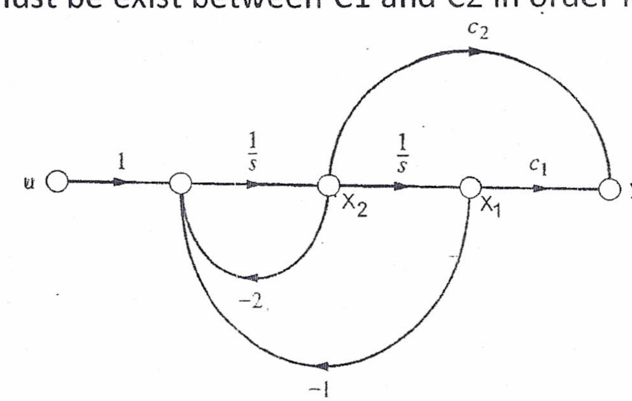
$$y = \begin{bmatrix} 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3. Determine whether or not each of the following systems is observable.





4. Given the plant below , what relationship must be exist between C1 and C2 in order for the system to be unobservable.

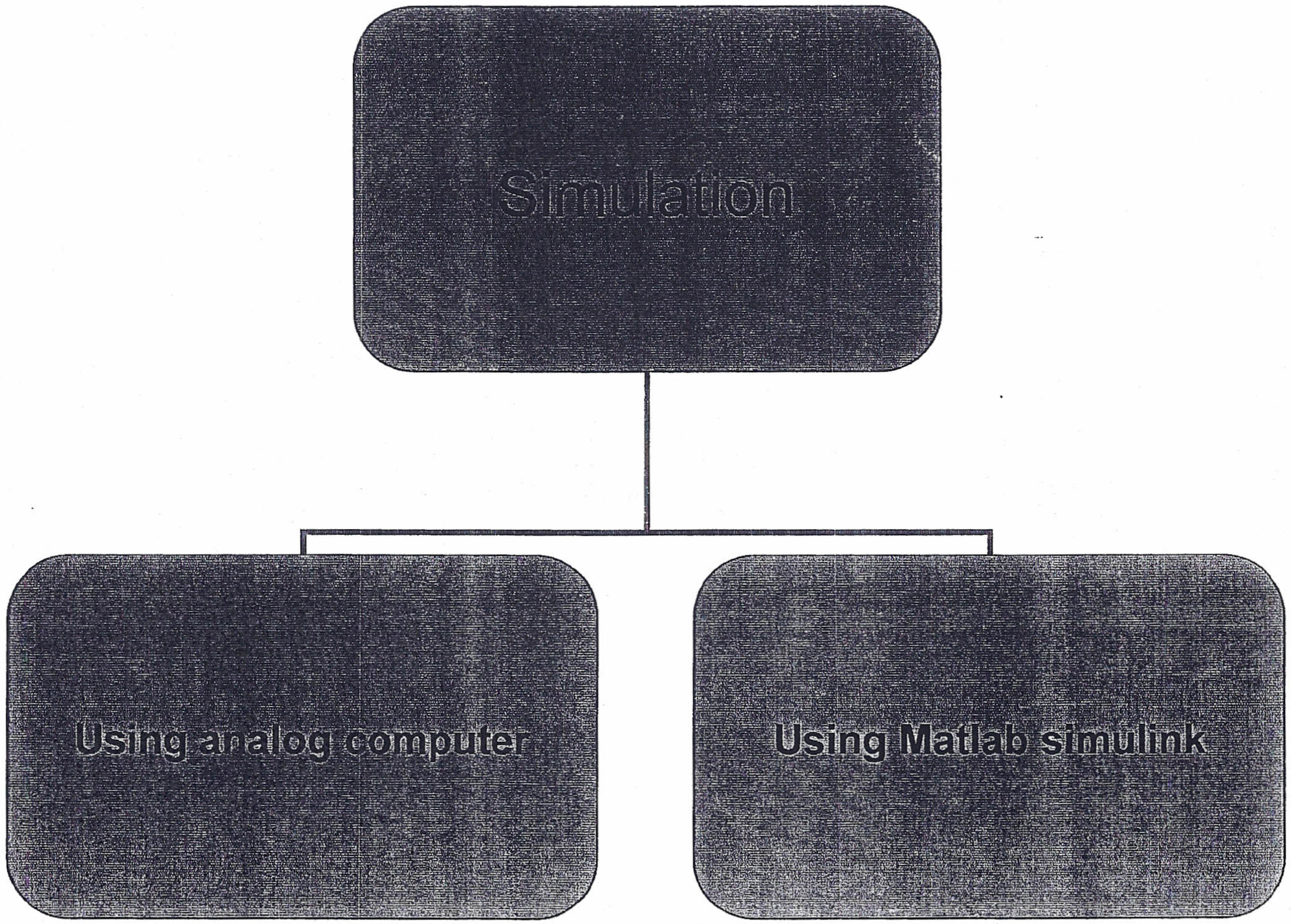


5- Transfer the following system into the observable canonical form

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Simulation

- **Simulation** : is a method to solve and find the response of complex systems which can not easily solved by analytical approach



Matlab simulink

- **Matlab simulation (software).** Simulink has become the most widely used software package in academia and industry for modeling and simulating dynamic systems.

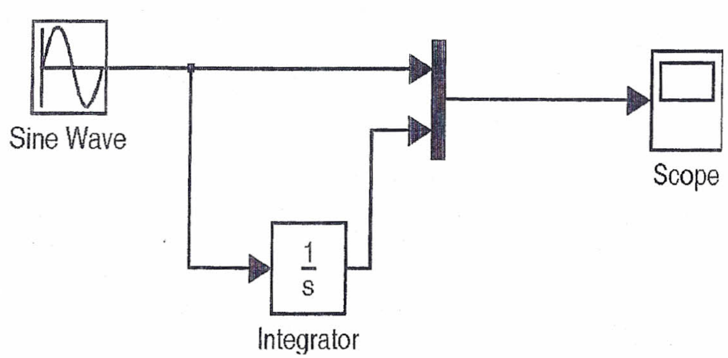
mathmatica

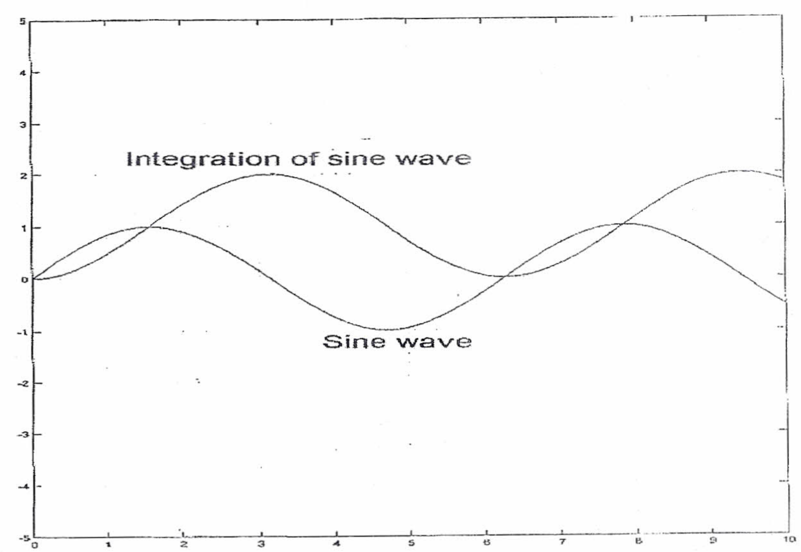
- Simulink is a software package for modeling, simulating, and analyzing dynamic systems. It supports linear and nonlinear systems, modeled in continuous time, sampled time.
- For modeling, Simulink provides a graphical user interface (GUI) for building models as block diagrams, using click-and-drag mouse operations. With this interface, you can draw the models easily.
- This is a far cry from previous simulation packages that require to formulate differential equations and difference equations in a language or program.

- Simulink includes a comprehensive block library of sinks, sources, linear and nonlinear components, and connectors.
- . After a model is defined , it can simulate it, using a choice of integration methods, either from the Simulink menus or by entering commands in the MATLAB Command Window.
- The menus are particularly convenient for interactive work, while the command-line approach is very useful for running a batch of simulations.
- Using scopes and other display blocks, the simulation results can be seen while the simulation is running.
- In addition, the parameters can be changed and immediately see what happens. The simulation results can be put in the MATLAB workspace for post-processing and visualization.

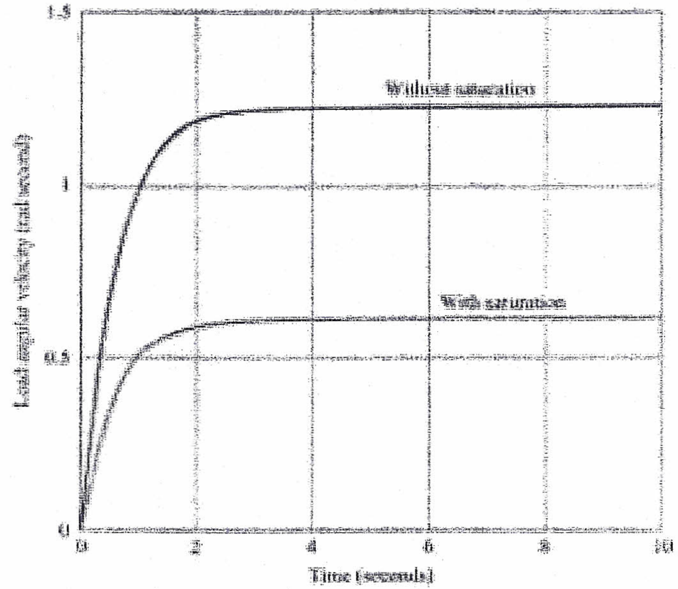
Examples

- How to model a simple system using simulink ?
- The model is to integrate a sine wave and displays the result along with the sine wave.
- To create the model, first enter simulink in the MATLAB command window. The Simulink Library Browser will appears.
- select the New Model button on the Library Browser's toolbar.
- Simulink opens a new model window.
- copy blocks into the model from the Simulink block libraries:
- The Sine Wave block from Sources library
- The Scope block from Sinks library.
- The Integrator block from Continuous library.
- The Mux block from Signals routing library.
- Then the system model is

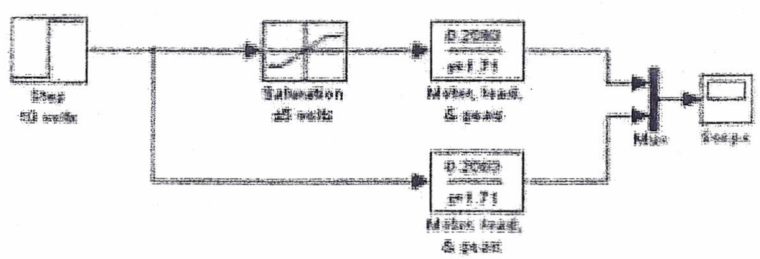




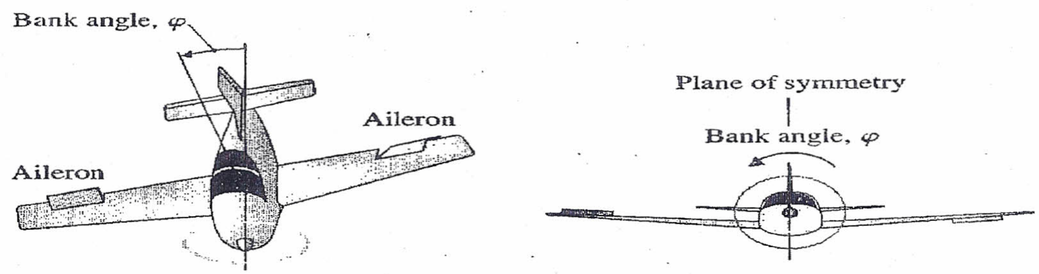
- Effect of amplifier saturation



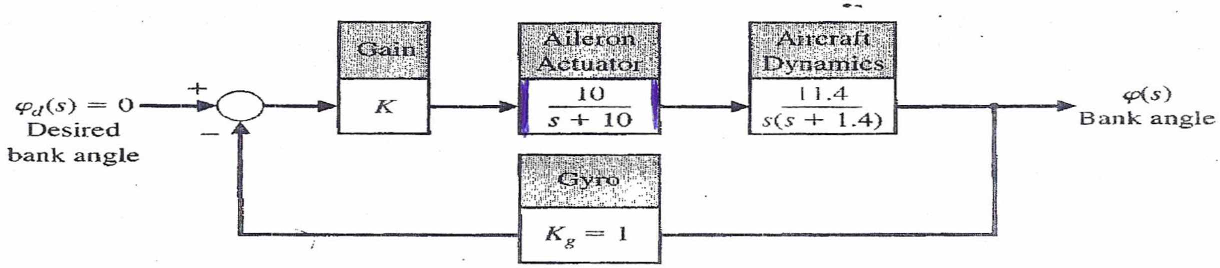
(a)



- Example 2
- For the following system draw the system simulink diagram and find the system response for different values of system parameters.

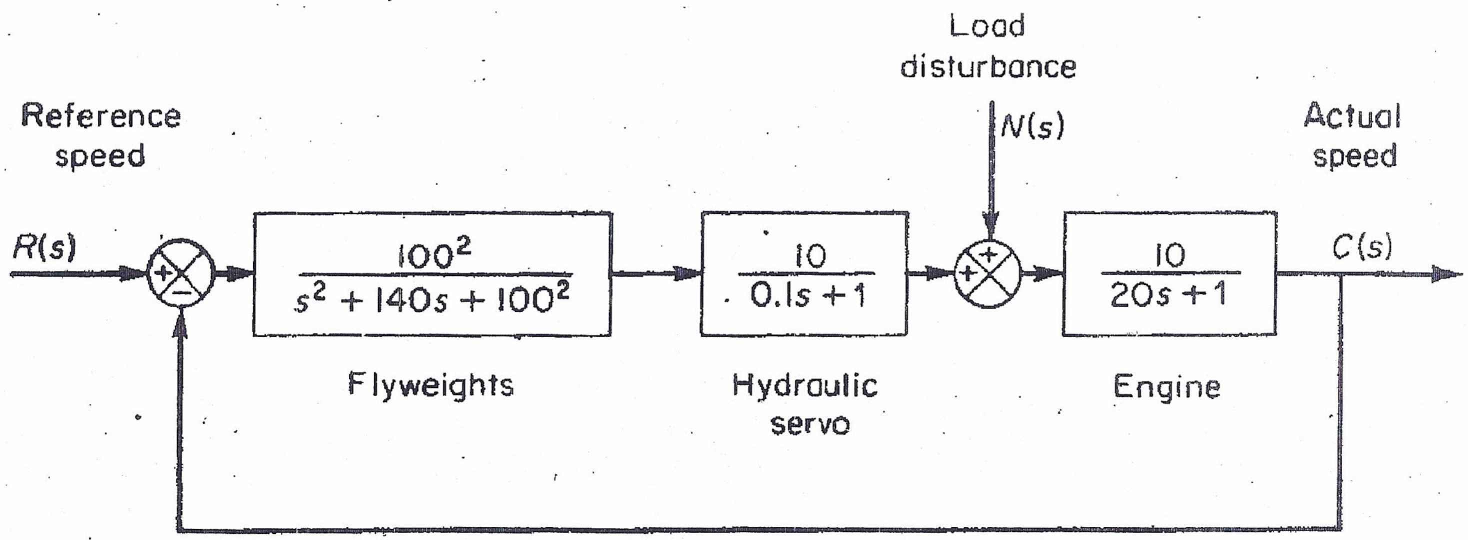


(a)

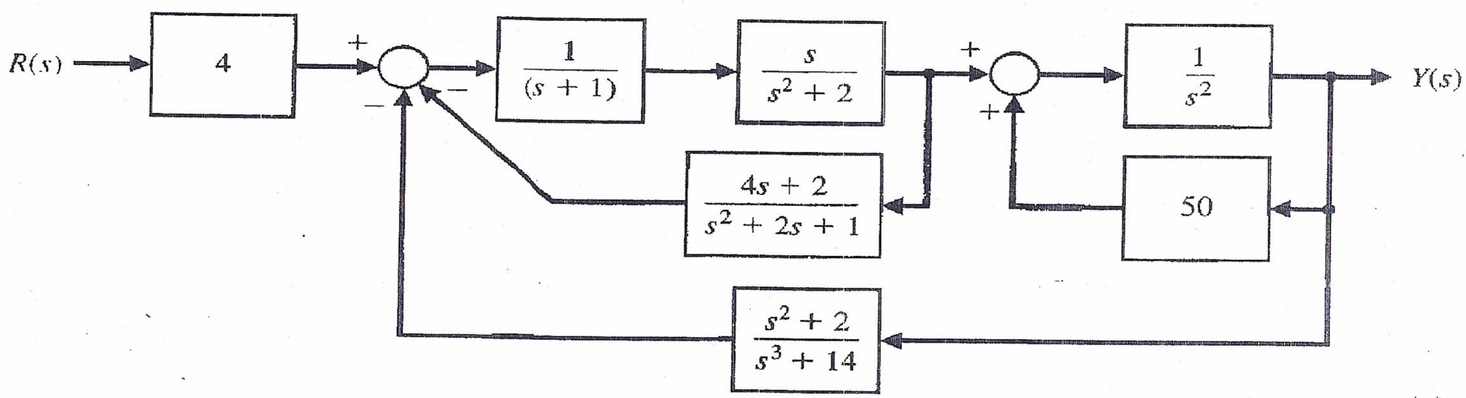


(b)

2.Show the effects of disturbance on the system response. Assume step input

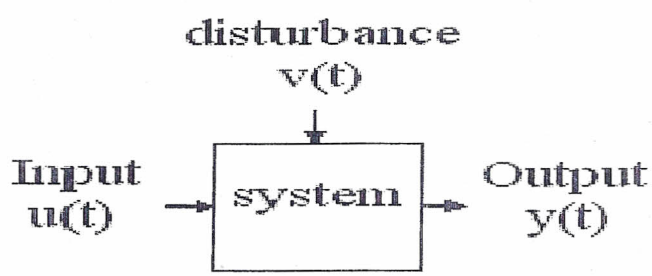


3. Find the response of the following system under step input.



System identification

- System identification is the field of modeling dynamic systems from experimental data.
- in many cases the data may be corrupted by measurement noise and/or disturbances, which aggravates the modeling process.
- the techniques of system identification have a wide application area,
- The system is driven by input variables and disturbances .The output signals are variables which provide useful information about the system..
- dynamic system can be described as in figure (1.1).

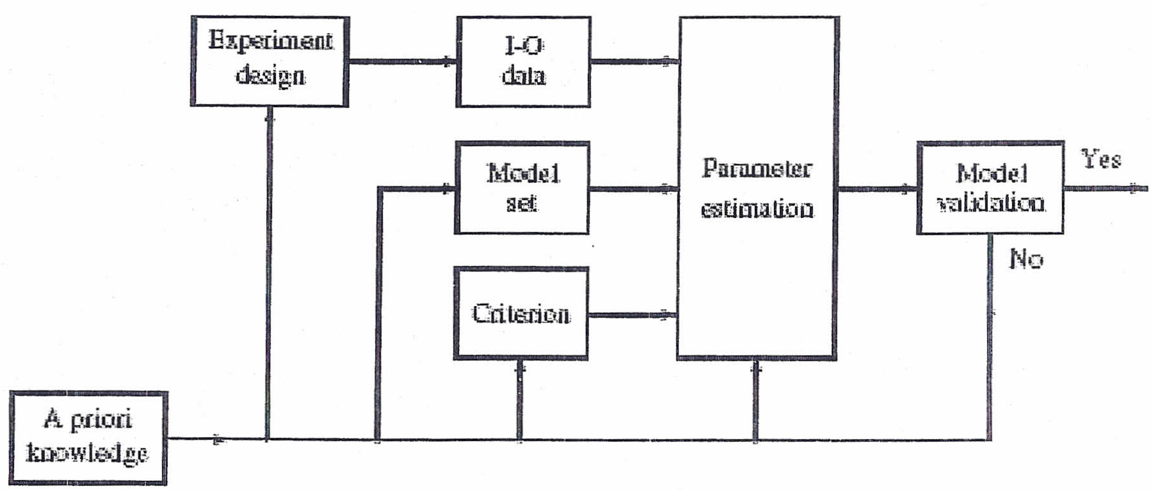


fig(1.1) A dynamic system

The basic information for System Identification Procedure

The construction of a model from data involves three basic entities:

- The input-output data
- The set of candidate models
- Determining the " best " model (model validation)



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- ## 1. The input/ output data:

- The input-output data are sometimes recorded during a specifically designed identification experiment, where the user may determine which signals to measure and when to measure them and may also choose the input signals. The object of experiment design is thus to make these choices so that the data become maximally informative, subject to constraints that may be at hand.

- In other cases the user may not have the possibility to affect the experiment, but must use Data from Normal operation of the system

2. The set of candidate models:

- a set of candidate models is obtained by specifying within which collection of models that are going to look for a suitable one.
- This is the most important, and at the same time, the most difficult choice of the system identification procedure .
- In many cases linear models are employed, without reference to the physical background of the system.
- Such a models set, whose parameters used to adjust and fit the data and do not reflect physical considerations in the system, is called a black box model.

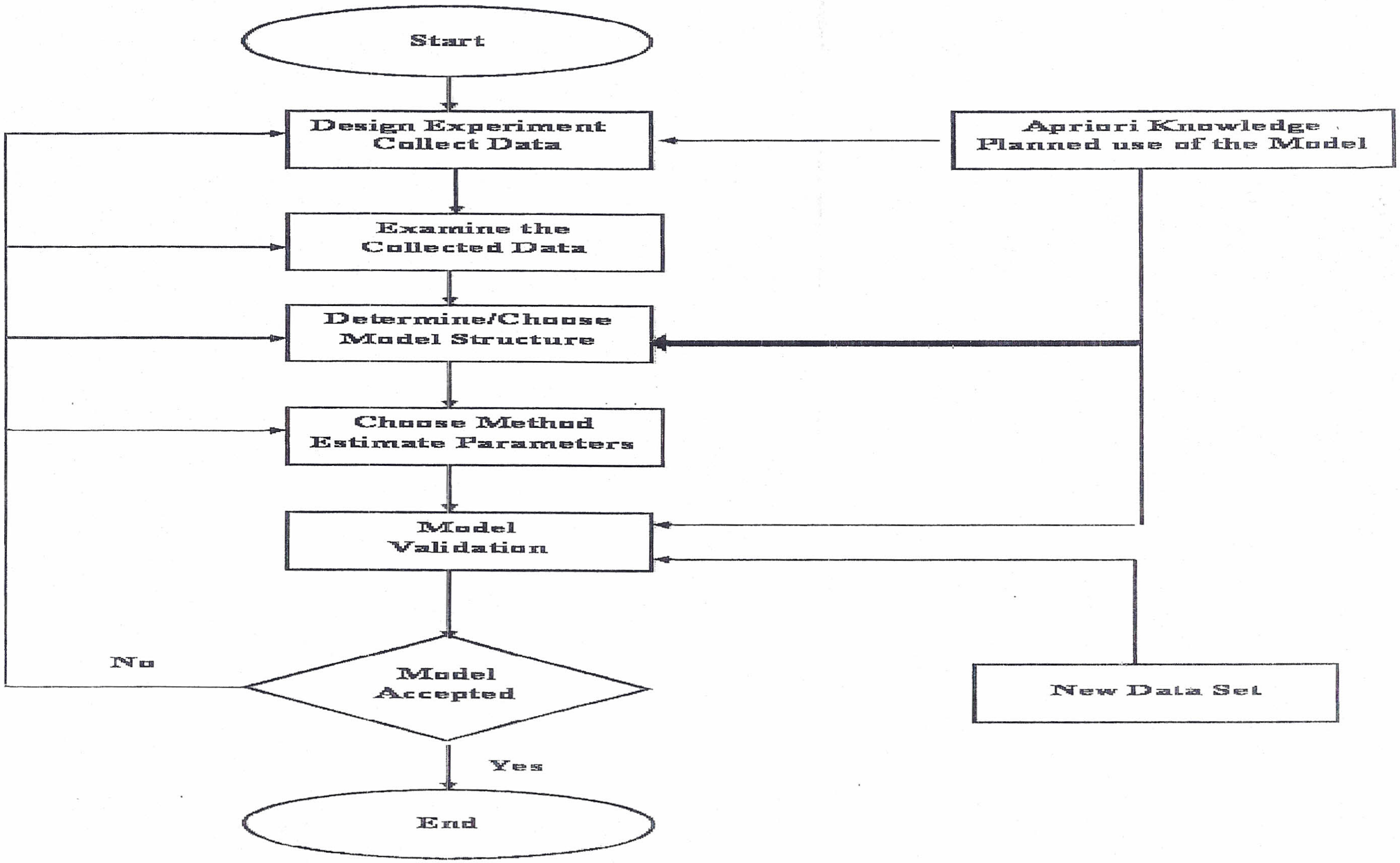
3. Determining the "best" model in the set (Model validation).

The assessment of model quality is typically based on how the models perform when they attempt to reproduce the measured data.
The system identification procedure is a recursive one. It is quite likely that the model first obtained will not pass the model validation tests. then must be go back and revise the various steps of the procedure.

Procedure of System Identification

- Design of Experiments
- Identification of Model Structure
- Parameter Estimation
- Model Validation

In the following figure the System Identification loop is shown:



The cycle can be itemized as follows

- Design an experiment and collect input-output data from the process to be identified.
- Examine the data. Polish it so as to remove trends and outliers, and select useful portions of the original data. Possibly apply filtering to enhance important frequency ranges.
- Select and define a model structure (a set of candidate system descriptions) within which a model is to be found.
- Compute the model parameters and then estimate the best model from the model structure according to the input-output data and a given criterion of fit.
- Examine the obtained model's properties by a validation data.
- If the model is good enough, then stop; otherwise go back to Step 3 to try another model set Possibly also try other estimation methods (Step 4) or work further on the input-output data (Steps 1 and 2).

Model Types

Different approaches to system identification depending on model class

- Linear /Non-linear
- Parametric/Non-parametric
  - **Non-parametric** methods try to estimate a generic model (step responses, impulse responses, frequency responses).
  - **Parametric** methods estimate parameters in a user specified model (transfer functions, state-space method).

Model descriptions:

- Transfer functions
- State-space models

Model Representation by Transfer-Functions:

The most immediate way of parameterize G and H is to represent them as rational functions and let the parameters be the numerator and denominator coefficients. Such model structures are also known as black-box models .

- ***ARX-MODEL Structure:***
  (Auto Regressive with exogenous input)

The simplest input-output relationship is obtained by describing it as a linear difference equation:

$$y(k)+a_1 y(k-1)+\ldots+a_{n_a} y(k-\mathrm{n_a})$$
$$=\mathrm{b_1} u(k-1)+\ldots+\mathrm{b_{n_b}} u(k-\mathrm{n_b})+\mathrm{e(k)}$$

Where:

$y(k-\mathrm{i})=$ Output in previous instant

$u(k-\mathrm{i})=$ Input in previous instant.

$n_a=$ Number of a coefficients.

$n_b=$ Number of b coefficients.

$e(k)=$ is the white-noise

The white-noise term $e(k)$ enters as a direct error in the difference equation.

Let:

$$\theta = [a_1 \ a_2 \ \dots \ a_{n_a} \quad b_1 \ b_2 \ \dots \ b_{n_b} \ ]^T$$

$$A(q) = 1 + a_1 q^{-1} + \dots + a_{n_a} q^{-n_a}$$

And

$$B(q) = b_1 q^{-1} + \dots + b_{n_b} q^{-n_b}$$

Then $G(q,\theta) = \dfrac{B(q)}{A(q)}$, $H(q,\theta) = \dfrac{1}{A(q)}$

The above model is called the ARX-MODEL, where AR refers to the autoregressive part $A(q)y(k)$ and X to the extra input $B(q)u(k)$.

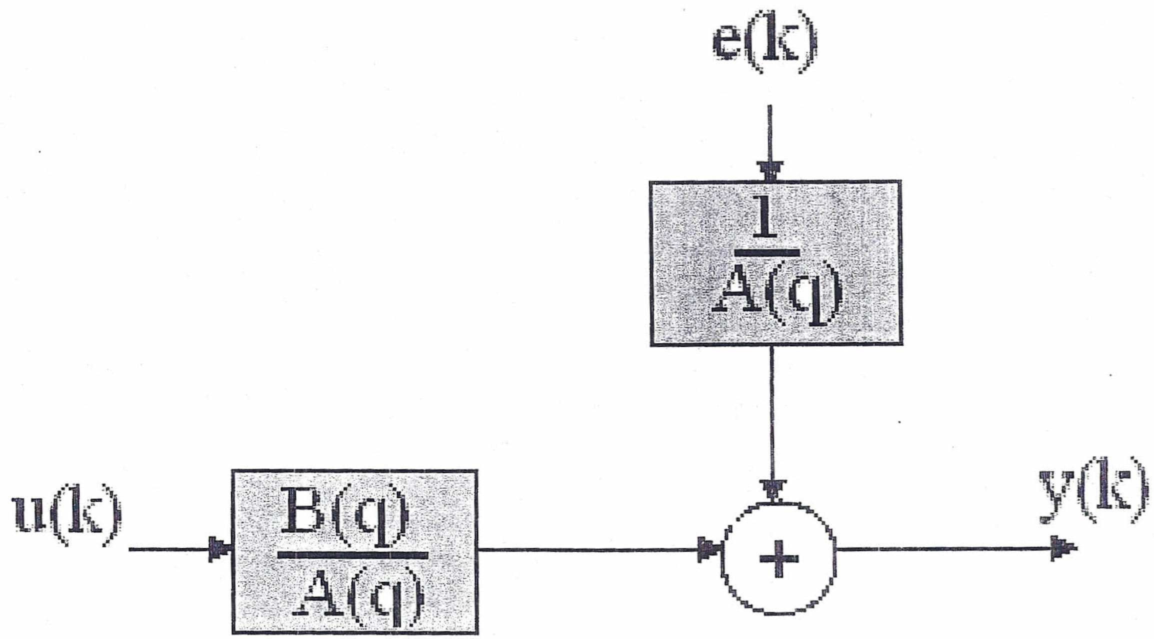


fig (2.2) The ARX model structure

•ARMAX-MODEL Structure:

(Auto Regressive Moving Average with exogenous input)

$$G(q,\theta)=\frac{B(q)}{A(q)}, \qquad H(q,\theta)=\frac{C(q)}{A(q)}$$

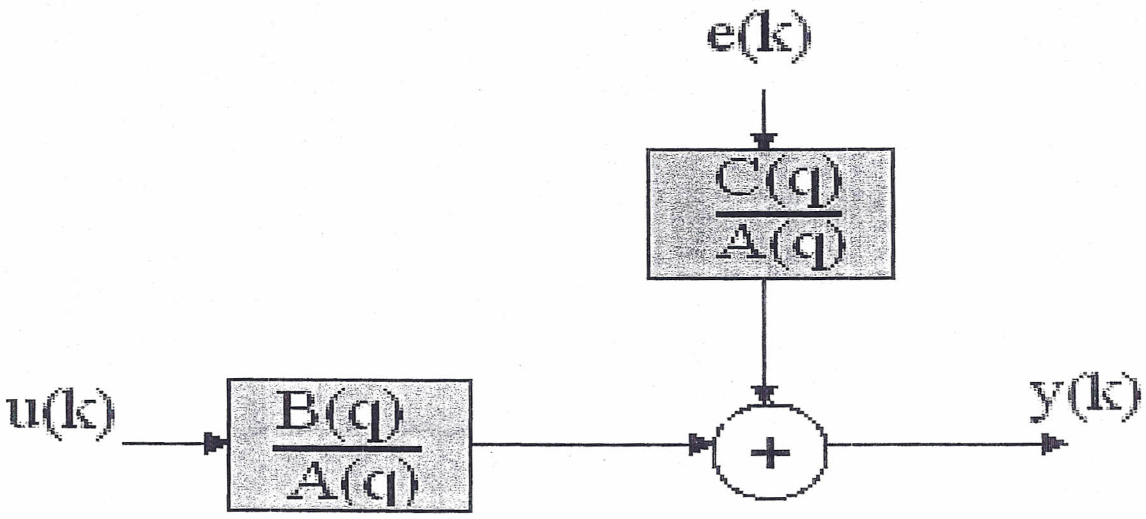


fig (2.3) The ARMAX model structure

• *OE-MODEL Structure:*
(Output Error model structure)

$$y(k) = \frac{B(q)}{F(q)} u(k) + e(k)$$

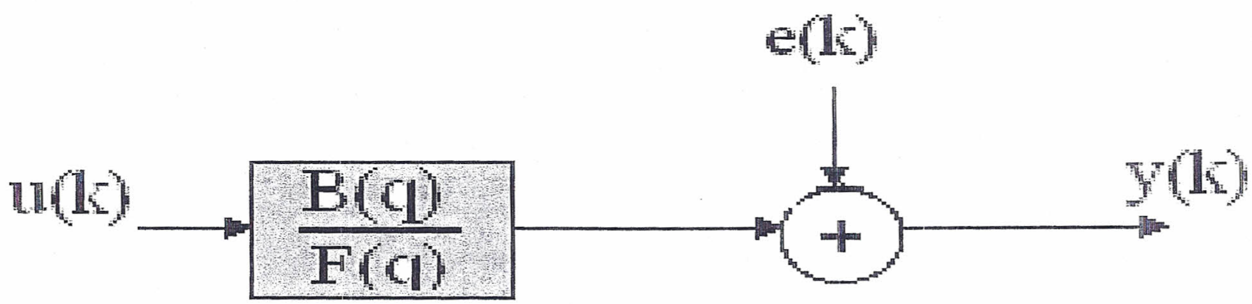


fig (2.4) The OE model structure

•*BJ-MODEL Structure:*
(Box-Jenkins model structure)

$$y(k)=\frac{B(q)}{F(q)}u(k)+\frac{C(q)}{D(q)}e(k)$$

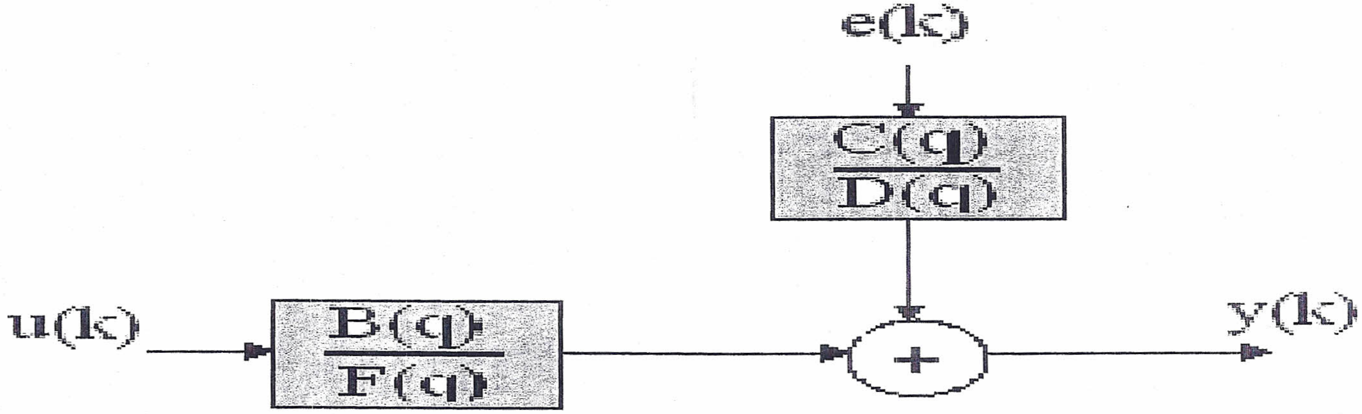


fig (2.5) The BJ model structure

Finally the general model structure could be displayed as in following equation:

$$A(q)y(k)=\frac{B(q)}{F(q)}u(k)+\frac{C(q)}{D(q)}e(k)$$

And depending on which of five polynomials A, B, C, D, and F are used it may give rise to 32 different model sets. And the simpler models which already discussed are a subset of the general model structure, and possible by setting one or more of $A(q), B(q), C(q), D(q), or\ F(q)$ polynomials equal to 1.

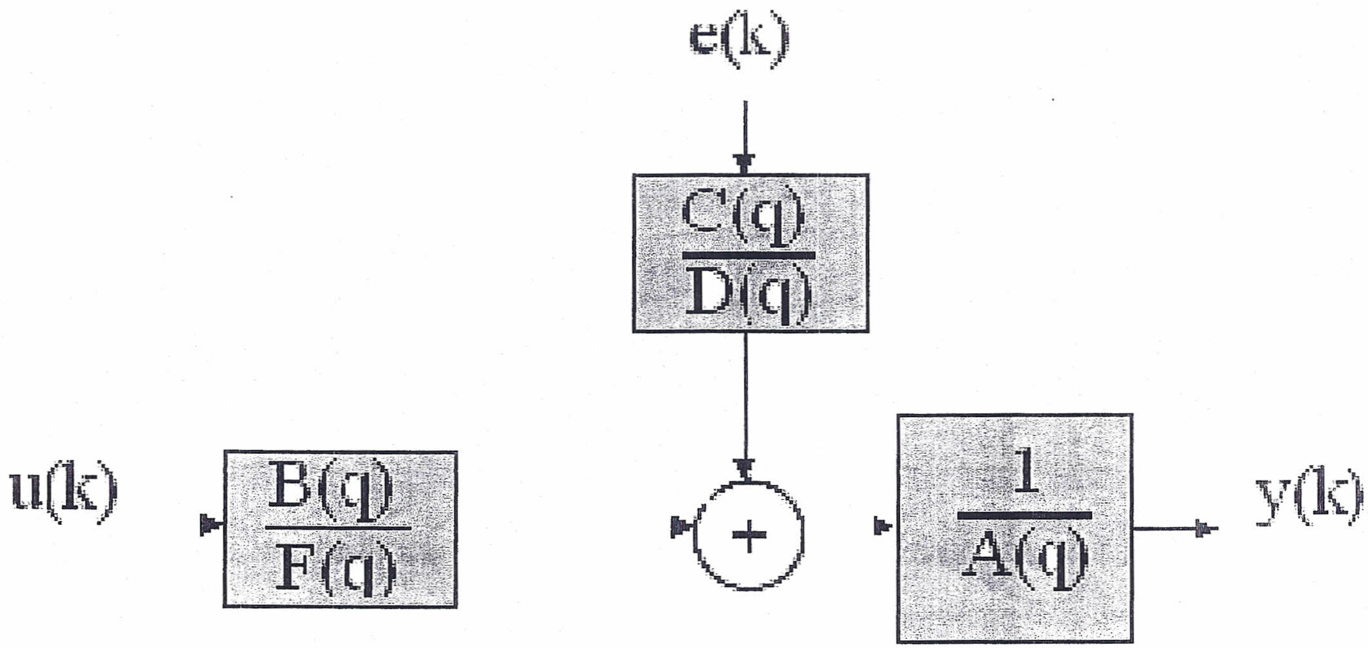


fig (2.6) The general model structure

Nonparametric Methods

This section describes some nonparametric methods for system identification; the nonparametric methods could be seen as preliminary experiments.

- **Transient response analysis:**
  - ***Step Response***
  - ***Impulse Response***

If the system which described by

$Y(t)= G(q)u(t)+v(t)$ eq 1

And

$$U(t) = \alpha \quad t \geq 0$$
$$= 0 \quad t < 0$$

Is applied to Eq. (1) gives the output.

$$y(t) = \alpha \sum_{k=0}^{t} g_0(k) + v(t) \qquad (2)$$

From this, estimates of $g_0(k)$ could be obtained as:

$$\hat{g}(t) = \frac{y(t) - y(t-1)}{\alpha} \qquad .3)$$

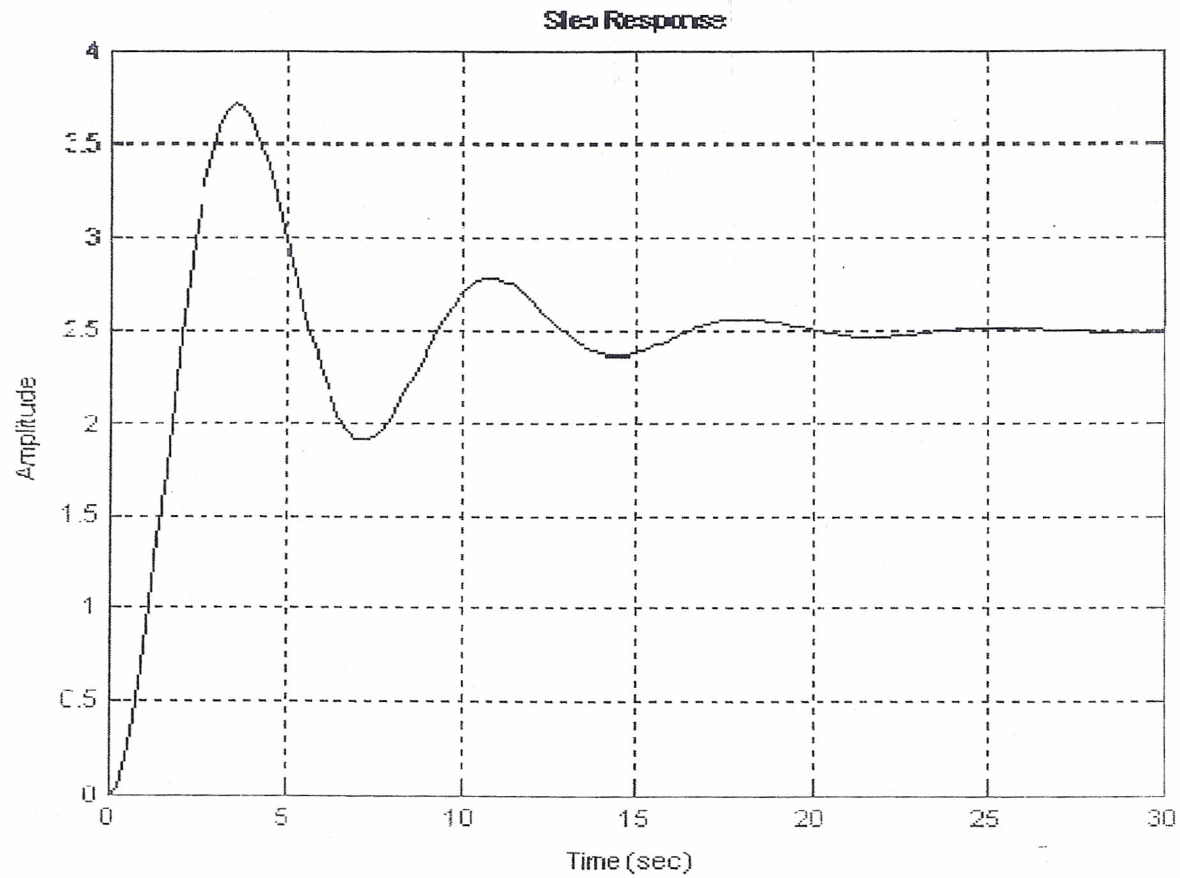
Which has an error $\dfrac{[v(t) - v(t-1)]}{\alpha}$.

The transient performance of a control system is usually characterized by the use of a unit step input.

And the typical performance criteria that are used to characterize the transient response to a unit step input are overshoot, delay time, rise time, settling time, static gain and time constant.

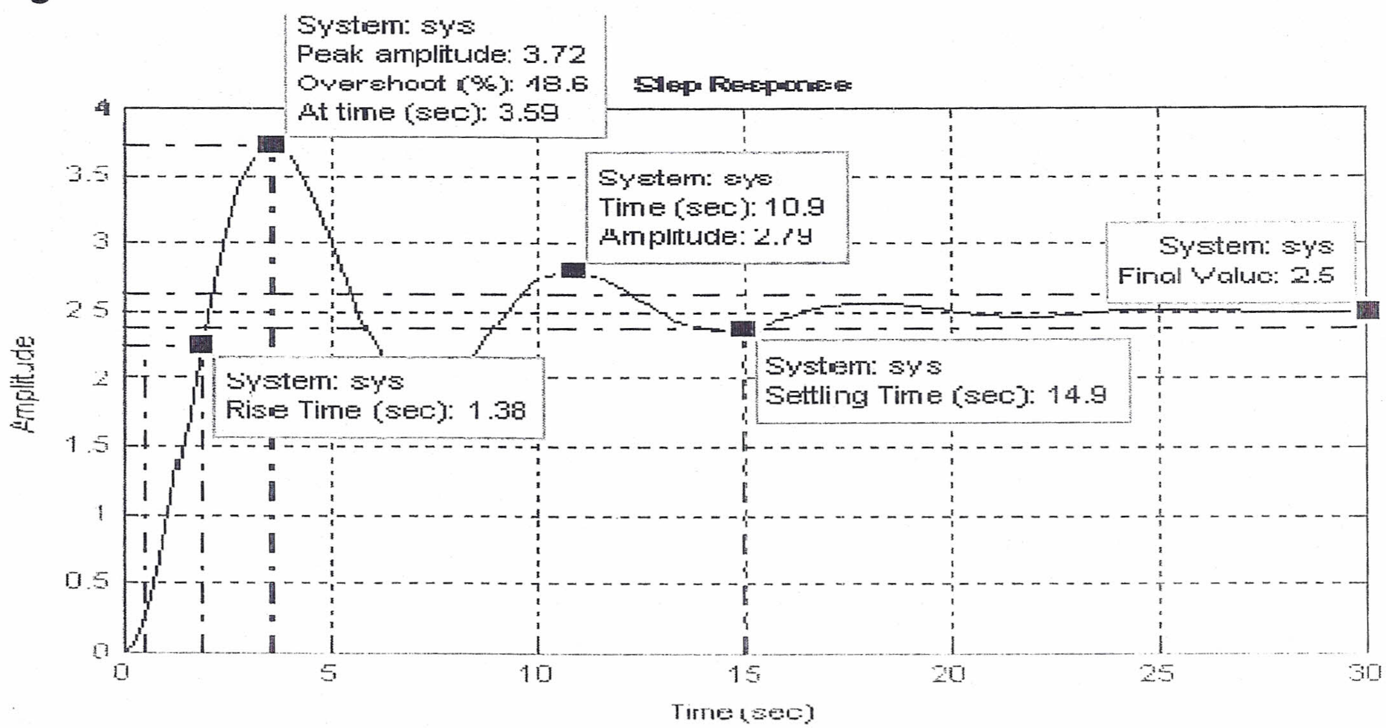
For example:

For step response of the system shown in fig (3.1) we will calculate all the step response parameters.



fig(3.1) step response

This system clearly is a second order or higher under damped system because the system oscillates.
The damping ratio of this system can be found using the percent overshoot. The percent overshoot can be measured of the step response by finding the ratio of the maximum achieved peak and the steady state output of the plot .
And by using the matlab we found that:



fig(3.2) step response parameters

•Frequency response analysis:

The input is a sinusoid, for a linear system in steady state the output will also be sinusoidal. The change in amplitude and phase will give the frequency response for the used frequency.

For a discussion of frequency response analysis it is convenient to use the continuous time model.

$$Y(s) = G(s)U(s) \tag{4}$$

If the input signal is a sinusoid

$$u(t) = a\sin(\omega t) \tag{5}$$

And the system is asymptotically stable, then in the steady state the output will become:

$$y(t) = b\sin(\omega t + \psi) \tag{6}$$

Where

$$b = a\left| G(j\omega) \right| \tag{7}$$

$$\psi = \arg\left| G(j\omega) \right| \tag{8}$$

For example:

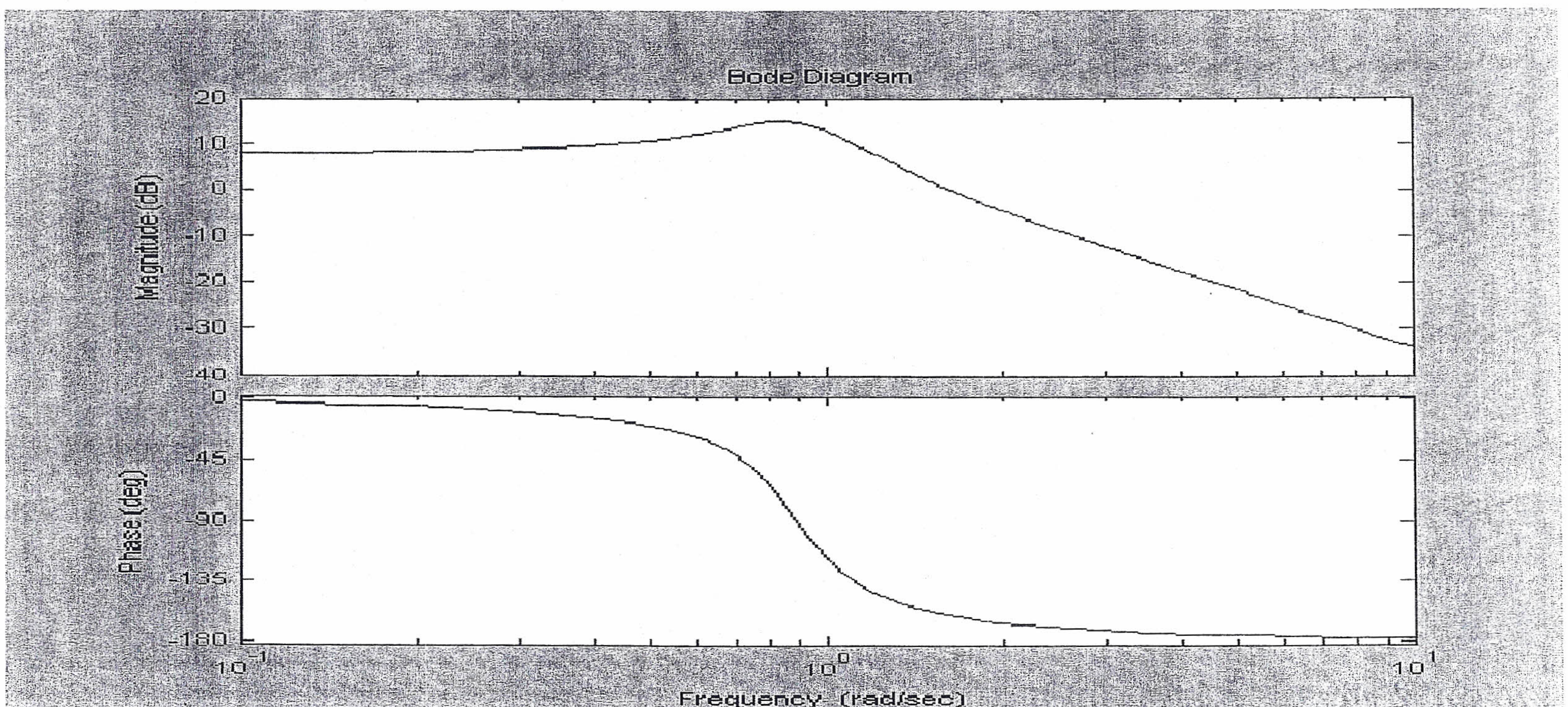


fig (3.3) Bode plot

For the bode plot fig(3.3) we will calculate the frequency response parameters as follows:

Estimating the Order

The phase plot of the bode plot dips asymptotically to -180 degrees relative to the input. -180 degrees is two multiples of -90 degrees, so the system is at least second order.

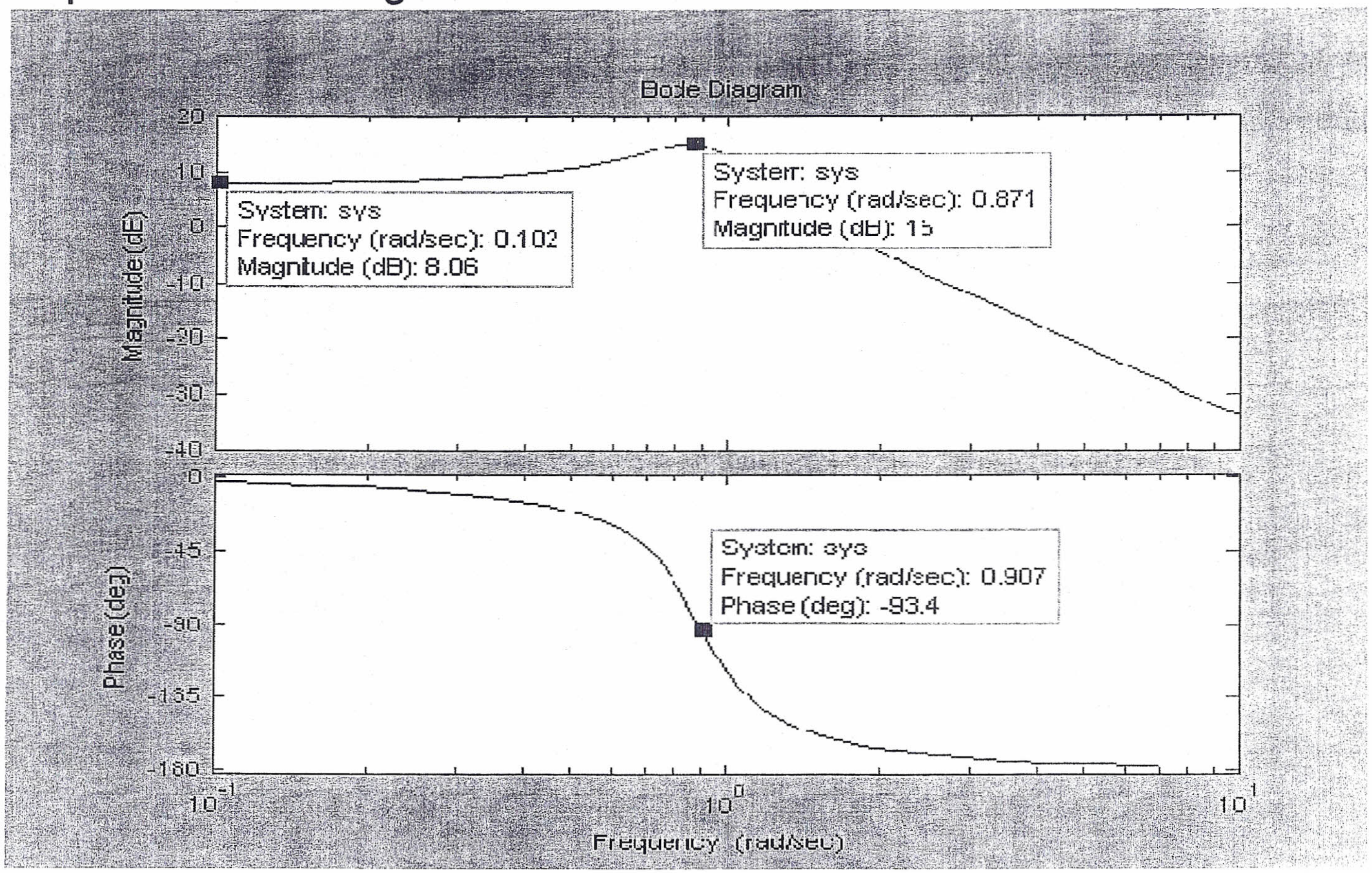
DC Gain

The DC gain of a system can be calculated from the magnitude of the bode plot when s=0.
The DC gain is:
DC Gain = 10^-(M0/20)
Where M0 is the magnitude of the bode plot when jω=0.
From the bode plot Fig(3.4):
M0 = 8.06
DC Gain = 0.4
The DC Gain of the system is 0.4

Natural Frequency

The natural frequency of a second order system is the frequency when the phase of the response is -90 degrees.

$$\omega_n = \omega_{-90^\circ}$$
$$= 0.907$$



fig(3.4)Bode plot parameters

Experiments and Data Collection

Often good to use a two-stage approach

- Preliminary experiments
  - Step/impulse response tests to get basic understanding of system dynamics
  - Linearity, static gains, time delays, time constants, sampling interval
- Data collection for model estimation
  - Carefully design experiment to enable good model fit
  - Operating point, input signal type, number of data points to collect

Preliminary Experiments – Step Response

Useful for obtaining a qualitative information about the system

- Dead time (delay)
- Static gain
- Time constants (rise time)
- Resonance frequency

Sample time can be determined from time constants

Choosing the Right Structure

- Starting with non-parametric estimates (correlation analysis, spectral estimation)
  - Gives information about model order and important frequency regions
- Prefilter input-output data to emphasize important frequency ranges
- Begin with ARX models
- Select model orders via
  - Cross-validation (simulate model and compare with new data)
  - Akiak's Information Criterion (AIC), i.e. pick the model that minimizes

$$\left(1 + 2\frac{d}{N}\right)\frac{1}{N}\sum_{t=1}^{N} \epsilon[t;\theta]^2$$

Where d is the number of estimated parameters in the model

Parametric Estimation Methods

- Non-recursive/Batch (off-line) methods
  - Linear regression and (blockwise) least squares
  - Prediction error methods
  - Instrumental variable methods
  - Subspace methods (if possible)
- Recursive (on-line) methods
  - Recursive Least Squares (RLS) methods
  - Recursive prediction error methods
  - Recursive instrumental variable methods
  - Forgetting factor techniques and time-varying systems
  - identification methods
  - Blockwise sliding window least squares methods

Principle of Model Validation

- Compare model simulation/prediction with real data *in time domain*
- Compare estimated model's frequency response and spectral analysis result *in frequency domain*
- Perform statistical tests on prediction errors

Keep Estimation and Validation Separate

- Split data into two parts
  - one for estimation
  - one for validation
- Apply input signal in validation data set to estimated model
- Compare simulated output with output stored in validation data set

Some commonly used input signals

- The input signal used in an identification experiment can have a significant influence on the resulting parameter estimation
  - Step function.
  - Pseudo Random Binary Sequence (PRBS).
  - With noise.
  - Sum of sinusoidal.

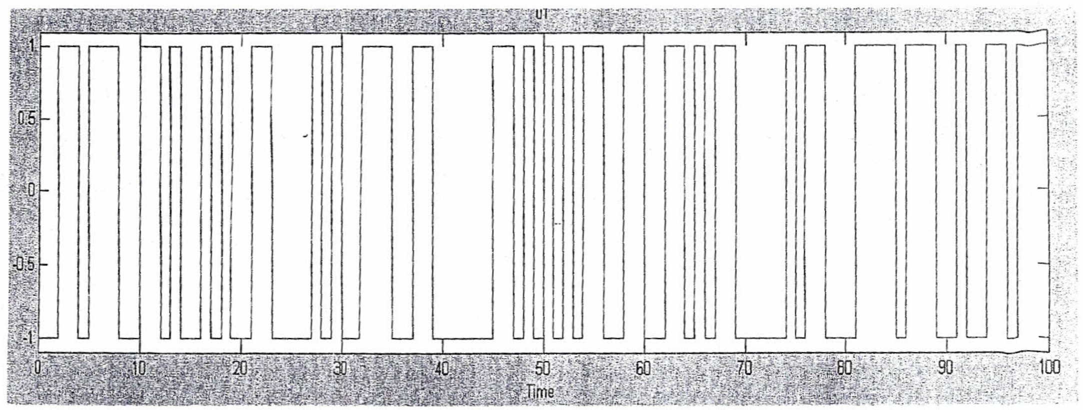
PRBS.

A pseudorandom binary sequence (PRBS) is a signal that shifts between two levels in a certain fashion. It can be generated by using shift registers for realizing a finite state system, and is a periodic signal.

In most cases the period is chosen to be of the same order as the number of samples in the experiment, or larger

When applying a PRBS, the user must select the two levels, the period and the clock period. The clock period is the minimal number of sampling interval

Let u(t) be a PRBS that shifts between the values a and $-a$, and let it's period be M, the autocorrelation function of PRBS is

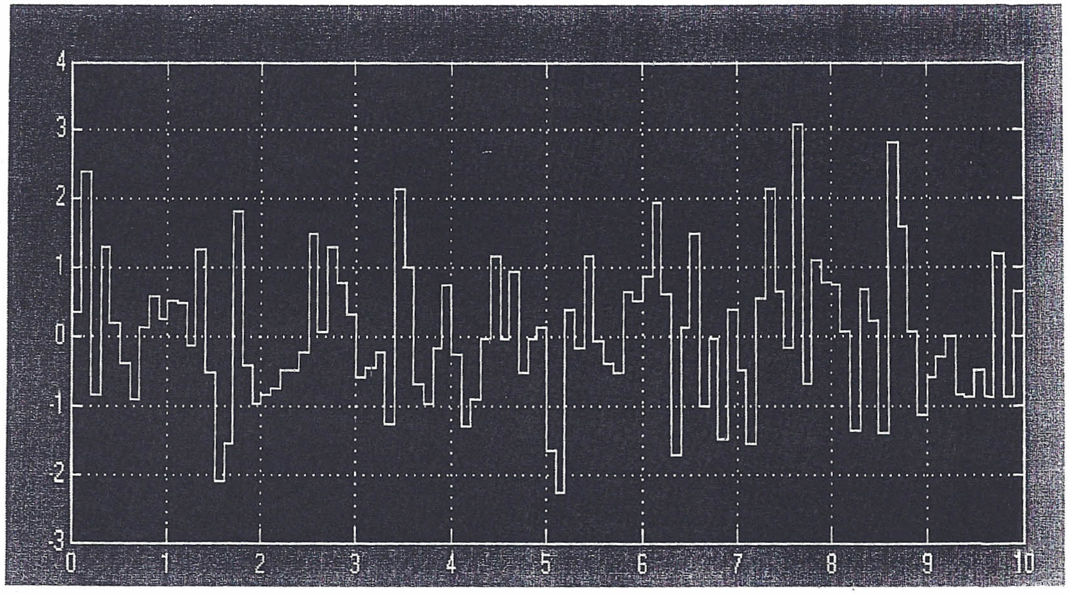


$$R_u(\tau) = \begin{cases} a^2 & \tau = 0, \pm M, \pm 2M, \dots\dots \\ -a^2/M & elsewhere \end{cases}$$

White noise

A sequence *u(k)* with autocorrelation function, zero except at $\tau = 0$, having constant power spectral density function is known as white noise .

The autocorrelation function of with noise is an impulse function



$$R_{uu}(\tau) = K\delta(\tau)$$

Correlation Analysis

- The correlation analysis method that estimates the impulse response is useful only when the input signal *u(k) is a zero-mean white noise signal. However,* the input signal is not white noise in most real-world applications. The input *u(k) and output y(k) signals therefore must be* preconditioned before you apply the correlation analysis method.

Correlation Analysis

- The correlation analysis method uses the cross correlation between the input and output signals as an estimation of the impulse response, as shown by the following equation:

$$y(k) = \sum_{k=0}^{\infty} u(k-n)h(n) + e(k)$$

- The input signal must be zero-mean white noise with a spectral density that is equally distributed across the whole frequency range.
- Assuming the input *u(k) of the system is a stationary, stochastic process* and statistically independent of the disturbance *e(k), the following equation* is true.

$$R_{uy}(\tau) = \sum_{k=0}^{\infty} R_{uu}(k-\tau)h(k)$$

- The above formula is called wiener-hopf equation.

Correlation Analysis

- $R_{uy}(\tau)$ represents the cross correlation function between the stimulus signal *u(k) and the response signal* y(k), as defined by the following equation.

$$R_{uy}(\tau) = \frac{1}{N} \sum_{k=0}^{N-\tau-1} y(k+\tau)\,u(k)$$

- $R_{uu}(\tau)$ represents the autocorrelation of the stimulus signal *u(k), as defined by* the following equation.

$$R_{uu}(\tau) = \frac{1}{N} \sum_{k=0}^{N-\tau-1} u(k+\tau)\,u(k)$$

- *N is the number of data points. If the stimulus signal is a zero-mean white* noise signal, the autocorrelation function reduces to the following equation.

$$R_{uu}(\tau) = \sigma_u^2 \delta(\tau)$$

Correlation Analysis

- where σ_u *is the standard deviation of the stimulus white noise and* $\delta(\tau)$ *is* the Dirac function. Substituting $R_{uu}(\tau)$ *into the cross correlation function* between the stimulus signal $u(k)$ *and the response signal* $y(k)$ *yields the* following equation.

$$R_{uy}(\tau) = \sigma_u^2 \sum_{k=0}^{\infty} \delta(k-\tau)\, h(k) = \sigma_u^2 h(\tau)$$

- There fore the impulse response h(k) *is proportional to the cross correlation function between* $u(k)$ *and* $y(k)$

$$h(k) = \frac{R_{uy}(k)}{\sigma_u^2}$$

Correlation Analysis

- If u(K) is PRBS signal. $R_{uu}(\tau) = a^2 \ell \delta(\tau)$

- Where ℓ is the clock pulse interval of PRBS.

Using wiener-hopf equation

$$R_{uy}(\tau) = a^2\ell \sum_{k=0}^{\infty} \delta(k-\tau)\, h(k) = a^2\ell h(\tau)$$

$$h(k) = \frac{R_{uy}(k)}{a^2\ell}$$

- Therefore the cross correlation function between u(k) and y(k) is proportional to the impulse response of the system h(k)

Remark

- Correlation analysis has a main advantages over classical methods of step and sine wave testing methods in that the plant is kept at normal production.
- The test input is superimposed with the operation input
- The impulse response is unaffected by normal operating input and the noise, provided that these signal are uncorrelated from the test input

Applications of the Impulse Response

- The impulse response not only indicates the stability and causality of the system if feedback exists in the system, but also provides information on properties such as the damping, dominating time constant, and time delay.
- Some of this information, such as the time delay, is useful for parametric model estimation. Therefore, you can use nonparametric impulse response estimation before parametric model estimation to help estimate the parameters.

Spectral Analysis Method

- The frequency response analysis method of obtaining the frequency response is straightforward but takes a long time to complete and is sensitive to noise. For these reasons, the spectral analysis method is used to estimate the frequency response function.
- You can use the spectral analysis method with any input signal. However, the frequency bandwidth of the input signal must cover the range of interest.
- Because the frequency response is the Fourier transform of the impulse response, applying the Fourier transform to both sides of the cross correlation function yields the following equation.

$$\Phi_{uy}\left(e^{j\omega}\right) = \Phi_{uu}\left(e^{j\omega}\right)G\left(e^{j\omega}\right)$$

Spectral Analysis Method

- $G(e^{j\omega})$ is the frequency response of the system.
- $\Phi_{uy}(e^{j\omega})$ is the auto-spectral density of the stimulus signal *u(k)*.
- $\Phi_{uu}(e^{j\omega})$ is the cross-spectral density between the stimulus signal *u(k)* and the response signal $y(k)$.
- You then can use the following equation to compute the frequency response

$$G(e^{j\omega})$$

$$G(e^{j\omega}) = \frac{\Phi_{uy}(e^{j\omega})}{\Phi_{uu}(e^{j\omega})}$$

Applications of the Frequency Response

- The frequency response gives the characteristics of the system in the frequency domain. You can use the frequency response to obtain useful information before applying parametric estimation. For example, you can use the frequency response to determine whether you need to pre-filter the signals or what the model order of the system is.
- You also can use nonparametric frequency response to verify parametric model estimation results by comparing the frequency response of the parametric model with the nonparametric frequency response.

Examples

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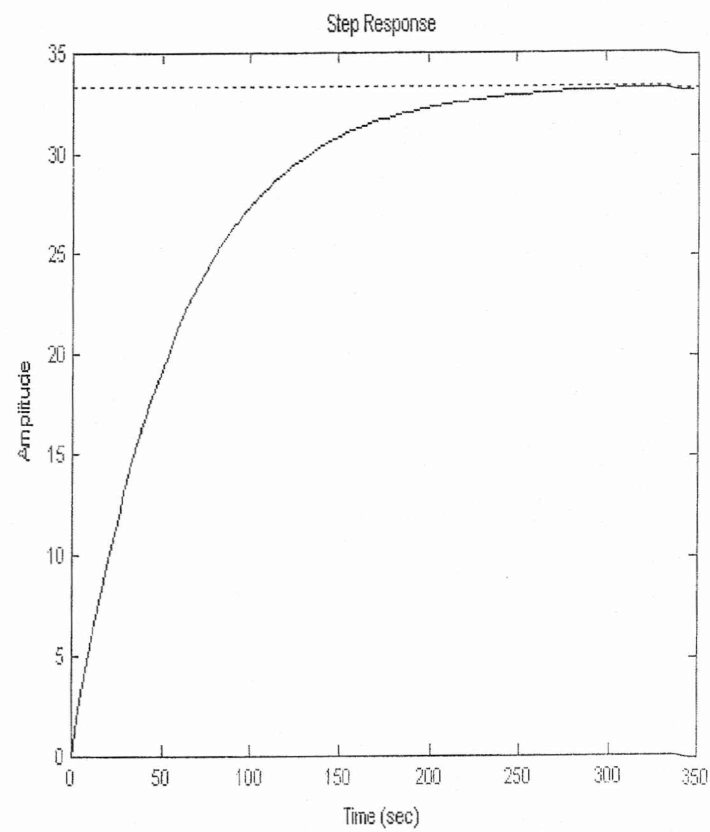
SYSTEM IDENTIFICATION of a first order system

Consider the truck with a different load in the trailer, the driver experiences a slower acceleration as seen in the following step response due to increased mass and increased friction due to the added load bearing down on the tires. Assuming the truck has the same engine which produces 20,000N of force



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SYSTEM IDENTIFICATION of a first order system

• ESTIMATING ORDER

This system appears to be a first order system, because the response does not oscillate and has a non-zero slope when $t = 0$. For this reason we will model this system as a first order system.

• DC GAIN

The DC gain is the ratio of the steady state step response to the magnitude of a step input. The steady state step response can be determined from the plot of the step response, and the magnitude of the step input is given as u=20000.

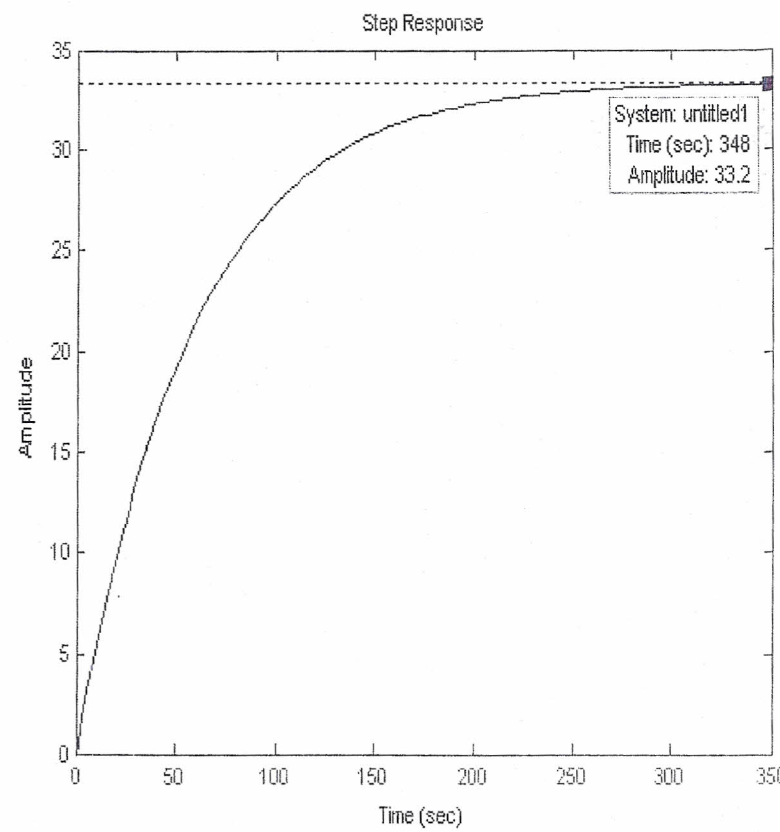
SYSTEM IDENTIFICATION of a first order system

The DC Gain is:

DC Gain = steady state output / step magnitude

```
u = 20000;
ss = 33.2;
K = ss/u
K =

    0.0017
```



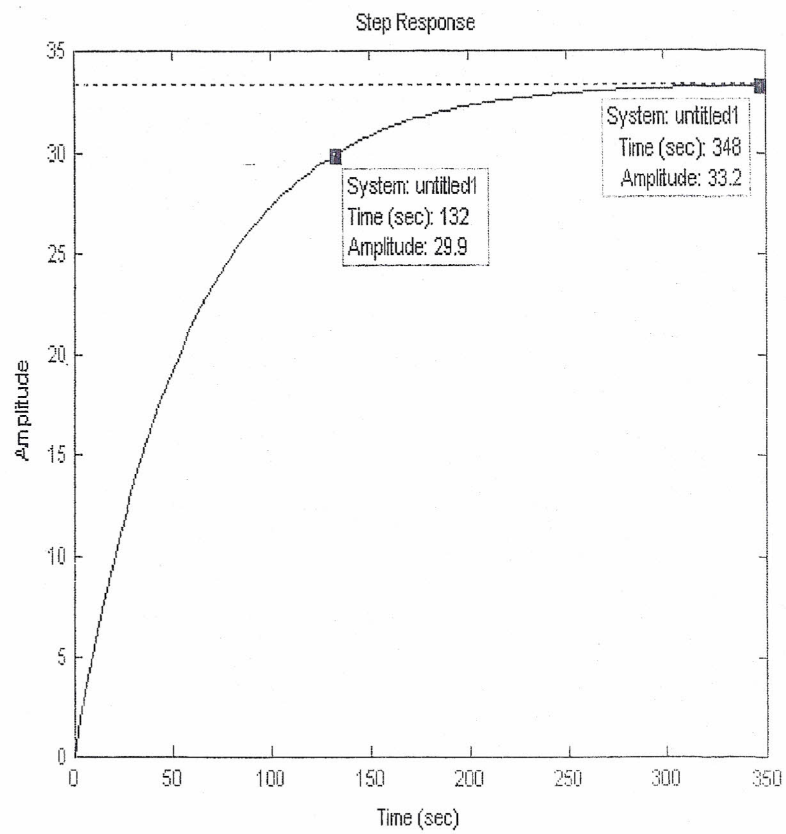
SYSTEM IDENTIFICATION of a first order system

SETTLING TIME

To find the settling time of the system, locate the time on the plot when the magnitude crosses the desired percentage of the final value. For instance to find the 10% settling time of this system, look for where the response reaches 0.9 of the final value:

```
0.9 * 33.2
ans =

    29.8800
```



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SYSTEM IDENTIFICATION of a first order system

- TIME CONSTANT

The time constant, t, of the system is the time at which the response is 1 - 1/e = 63% of the final value. The relationship between the time constant and the pole of a system is:

pole = -1 / t. Other handy approximate relations for finding the time constant are:

t = 10% Settling Time / 2.3
t = 5% Settling Time / 3
t = 2% Settling Time / 4

SYSTEM IDENTIFICATION of a first order system

- Because the 10% settling time is 132 seconds, the time constant is: t = 10% Settling Time / 2.3 = 132s / 2.3 = 57s
- The time constant of this system is approximately 57 seconds.
- It is also possible to find the time constant of the system by following the procedure for finding the settling time except for a percentage of 37%.

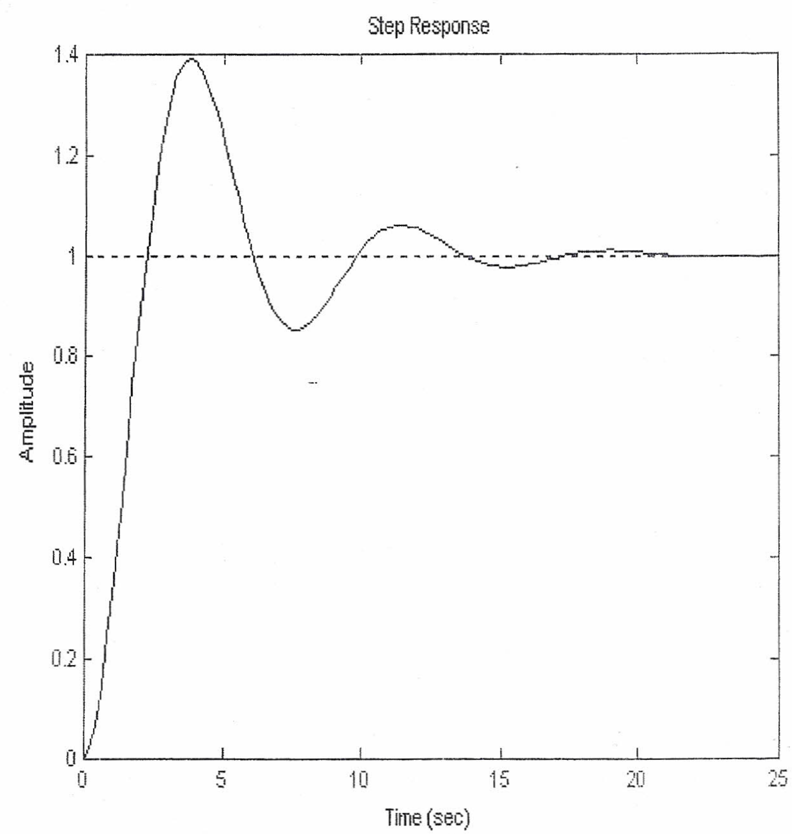
SYSTEM IDENTIFICATION of the second order system

UNDERDAMPED EXAMPLE

Identify the system with the following response to a step input of magnitude 3.

ESTIMATING ORDER

This system clearly is a second order or higher underdamped system because the system oscillates. The step response also has a zero initial slope when t=0, reinforcing the conclusion that the system is second order or higher. For this reason we will model this system as an underdamped second order system.



SYSTEM IDENTIFICATION of the second order system

DAMPING RATIO

The damping ratio of this system can be found using the percent overshoot. The percent overshoot can be measured off the step response by finding the ratio of the maximum achieved peak and the steady state output Off the plot, the percent overshoot is:

%OS = 100% (peak value - steady state output) / steady state output

peak = 1.39;

ss = 1;

os = 100*(peak-ss)/ss

os = 39

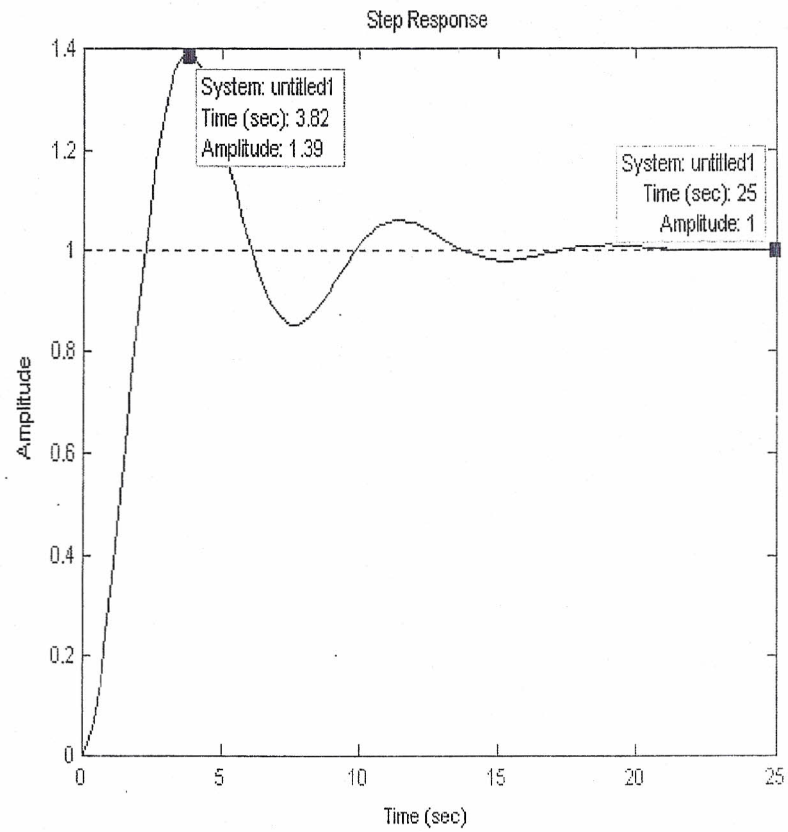
The damping ratio is:

$z = -\ln(\%OS/100) / \mathrm{sqrt}(pi^2+\ln^2(\%OS/100))$

dampingratio = -log(os/100)/sqrt(pi^2+(log(os/100))^2)

dampingratio = 0.2871

The damping ratio is 0.29, which is less than one, so this system is underdamped.



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SYSTEM IDENTIFICATION of the second order system

DC GAIN

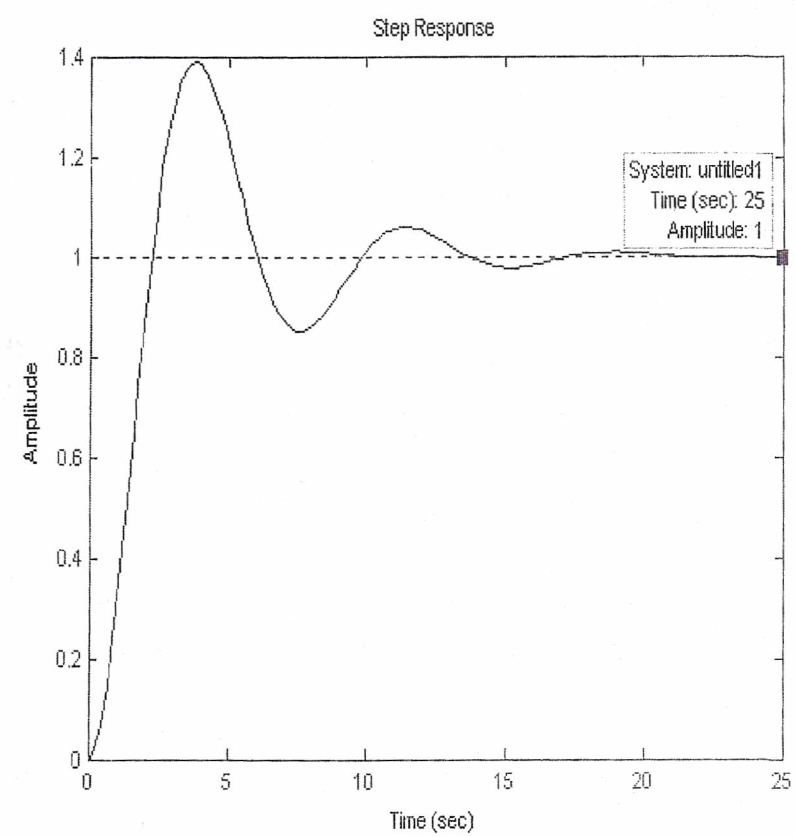
The DC gain is the ratio of the steady state step response to the magnitude of a step input. The steady state step response can be determined from the plot of the step response, and the magnitude of the step input is given as u=3. The DC Gain is:

DC Gain = steady state output / step magnitude

```
u = 3;
ss = 1;
dcgain = ss/u
dcgain =

      0.3333
```

The DC Gain of this system is 0.33.



SYSTEM IDENTIFICATION of the second order system

NATURAL FREQUENCY

The natural frequency of an underdamped second order system can be found from the damped natural frequency which can be measured off the plot of the step response and the damping ratio which was calculated above. The natural frequency is:

$w_n = w_d / \text{sqrt}(1 - z^2)$

The damped natural frequency is:

$w_d = 2p/Dt$

where D*t* is the time interval between two consecutive peaks on the plot of the step response.

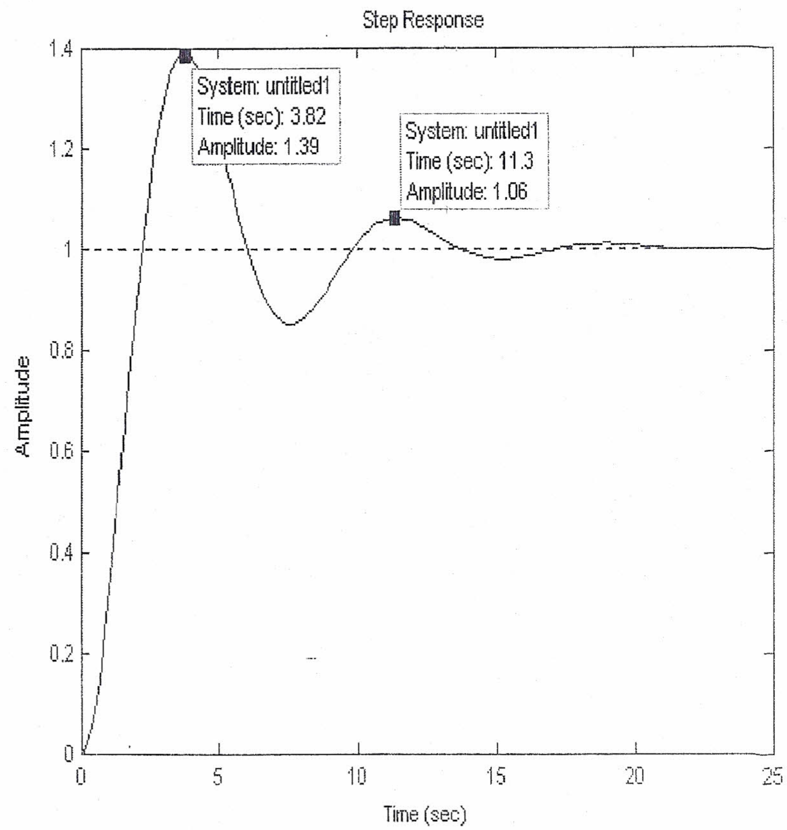
dt = (11.3 - 3.82);

wd = 2*pi/dt;

wn = wd/sqrt(1-dampingratio^2)

wn = 0.8781

The natural frequency of this system is approximately 0.88 rad/s which means that in the absence of all damping, this system will oscillate at 0.88 rad/s.

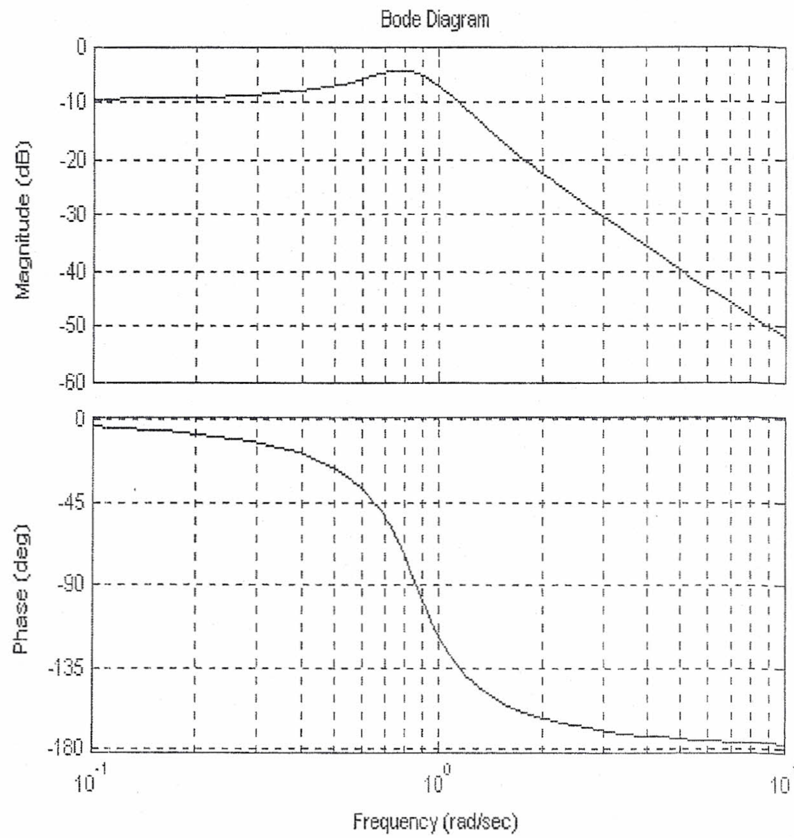


SYSTEM IDENTIFICATION of the second order system

Identify the following system with the Bode Plot.

ESTIMATING ORDER

The phase plot of the bode plot dips asymptotically to -180 degrees relative to the input. -180 degrees is two multiples of -90 degrees, so the system is at least second order.



SYSTEM IDENTIFICATION of the second order system

DC GAIN

The DC Gain of a system can be calculated from the magnitude of the bode plot when s=0.

The DC Gain is:

DC Gain = $10^{M(0)/20}$

where M(0) is the magnitude of the bode plot when jw=0.

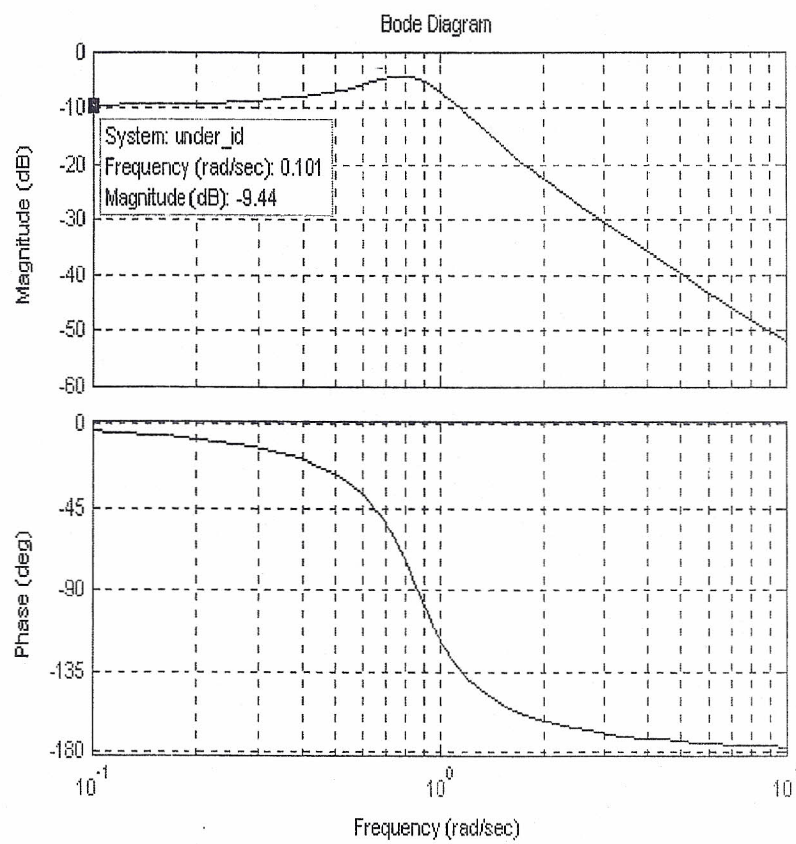
Using MATLAB:

M0 = -9.44;

dcgain = 10^(M0/20)

dcgain = 0.3373

The DC Gain of the system is 0.34.



SYSTEM IDENTIFICATION of the second order system

Natural Frequency

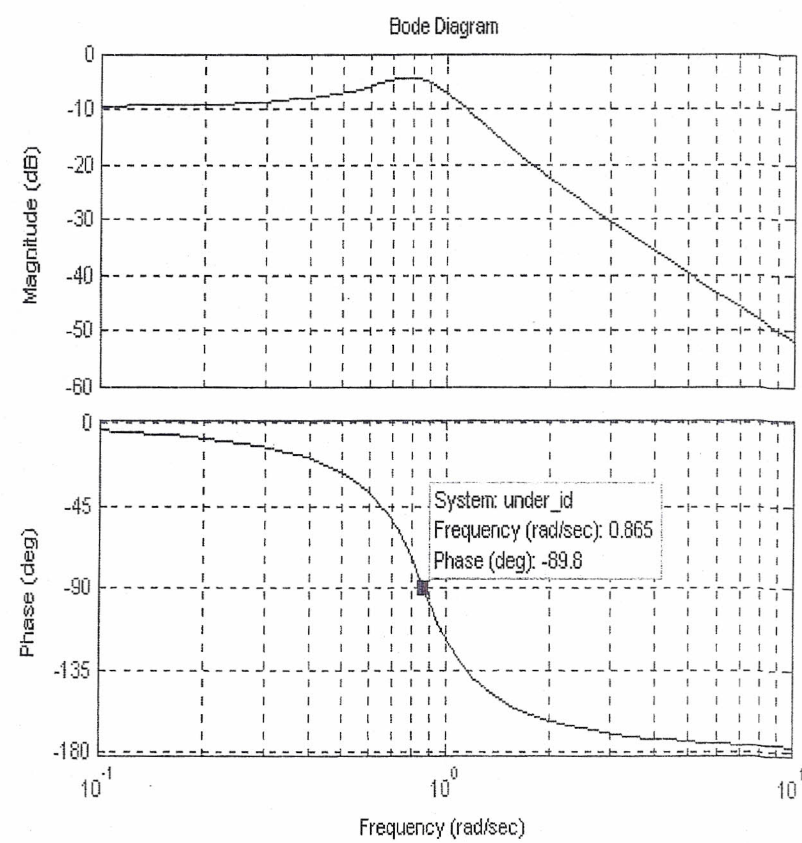
The natural frequency of a second order system is the frequency when the phase of the response is -90 degrees.

The natural frequency is:

$w_n = w_{-90°}$

$= 0.865$

where $w_{-90°}$ is the frequency at which the phase plot is at -90 degrees.



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SYSTEM IDENTIFICATION of the second order system

DAMPING RATIO

The damping ratio of a system can be found with the DC Gain and the magnitude of the bode plot when the phase plot is -90 degrees. The damping ratio is:

$z = K / (2*10^{(M\text{-}90°/20)})$

where $M_{-90°}$ is the magnitude of the bode plot when the phase is -90 degrees.

Using MATLAB:

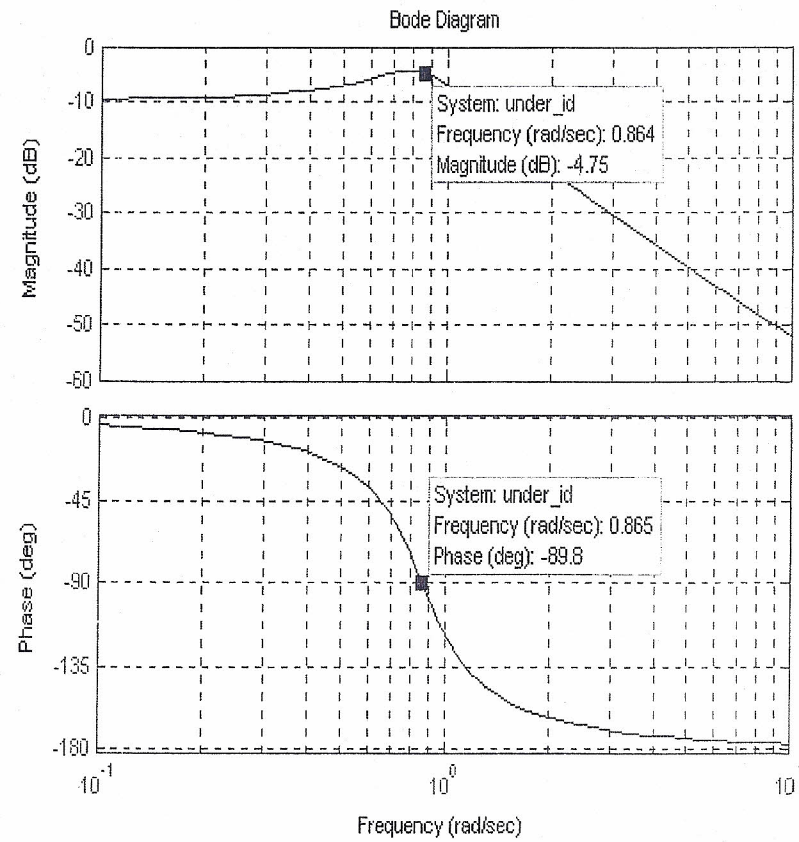
M90 = -4.75;

dampingratio = dcgain/(2*10^(M90/20))

dampingratio =

0.2914

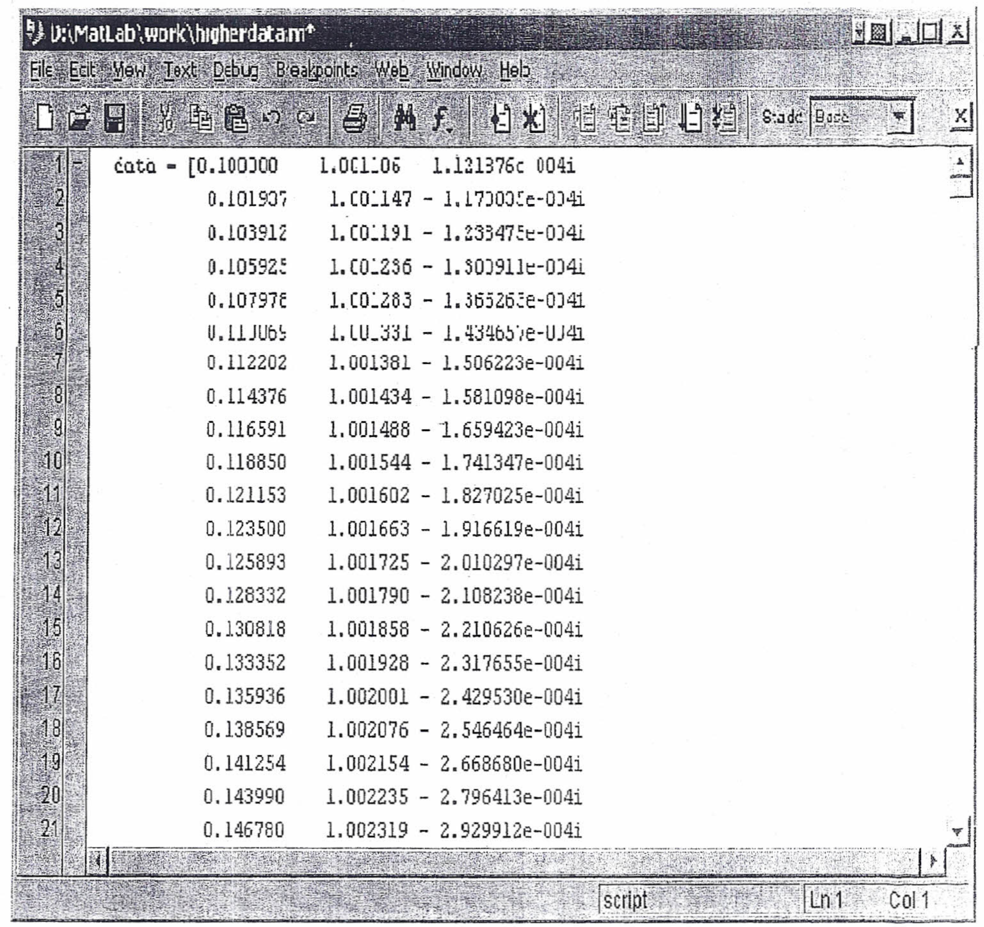
The damping ratio of this system is 0.29 which is less than one, so this system is underdamped.



2-1

SYSTEM IDENTIFICATION of a high order system

Higher order systems tend to be difficult to model because, in general, there are no analytical formulas to compute the parameters of a model directly from the experimental data. However it is often possible to break down a modeling problem into the problems of deriving the parameters of first and second order systems. In this example, the Bode plot for the frequency response of a given set of experimental data will be analyzed similar to the previous examples but with unknown transfer function and steady state parameters




```
data = [0.100000    1.001106   1.121376e 004i
        0.101907    1.001147 - 1.170035e-004i
        0.103912    1.001191 - 1.233475e-004i
        0.105925    1.001236 - 1.300911e-004i
        0.107978    1.001283 - 1.365263e-004i
        0.110069    1.001331 - 1.434657e-004i
        0.112202    1.001381 - 1.506223e-004i
        0.114376    1.001434 - 1.581098e-004i
        0.116591    1.001488 - 1.659423e-004i
        0.118850    1.001544 - 1.741347e-004i
        0.121153    1.001602 - 1.827025e-004i
        0.123500    1.001663 - 1.916619e-004i
        0.125893    1.001725 - 2.010297e-004i
        0.128332    1.001790 - 2.108238e-004i
        0.130818    1.001858 - 2.210626e-004i
        0.133352    1.001928 - 2.317655e-004i
        0.135936    1.002001 - 2.429530e-004i
        0.138569    1.002076 - 2.546464e-004i
        0.141254    1.002154 - 2.668680e-004i
        0.143990    1.002235 - 2.796413e-004i
        0.146780    1.002319 - 2.929912e-004i
```

SYSTEM IDENTIFICATION of a high order system

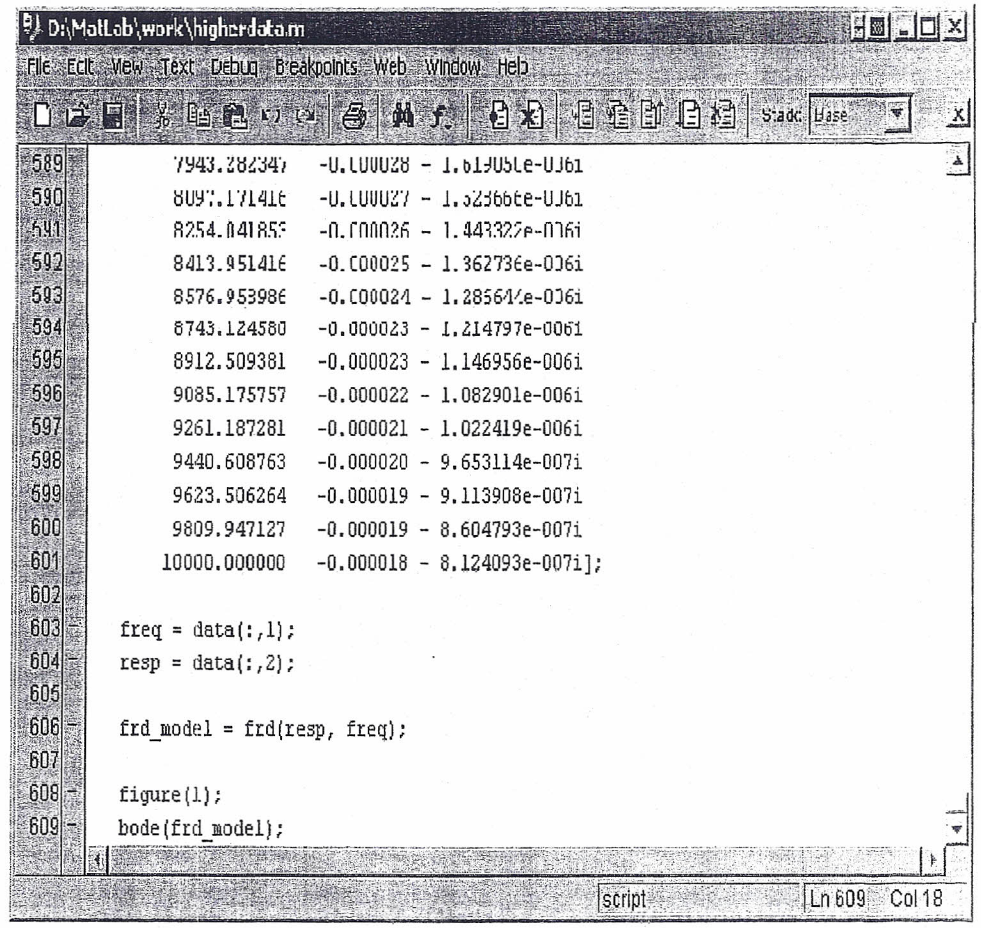
The data must now be turned and stored as a Frequency Response Data (FRD) model. To store the data as a FRD model, the frequency and response must be stored separately with the following commands“

freq = data(:,1);

resp = data(:,2);

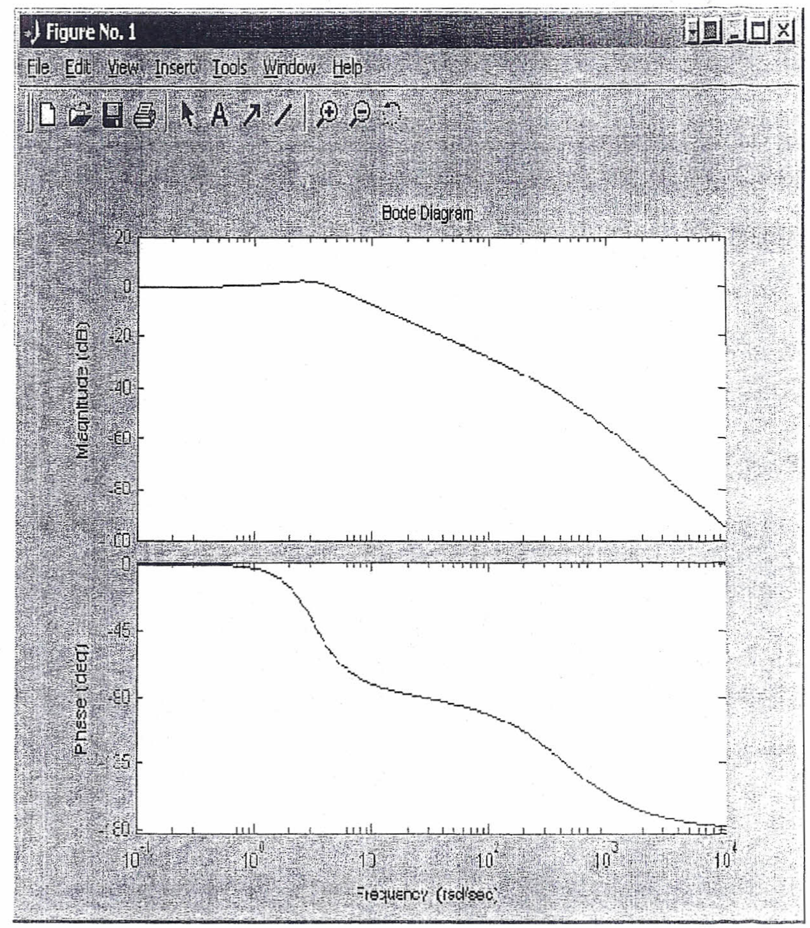
Now that the frequency and corresponding response are separate, they can be stored using the FRD command.

frd_model = frd(resp, freq);



SYSTEM IDENTIFICATION of a high order system

With this model we will use the Bode plot of the FRD model to create and compare a second order system and its corresponding Bode plot. The following will be the resulting Bode plot. Notice on the above magnitude plot the fairly level graph from 0.1 to 3 followed by a peak and a drop. At high frequencies, the magnitude graph decreases at about 40 dB/dec. Also, notice that the phase plot decreases from 0 to -180. These are all characteristics of a second order system. So to start the modeling process, construct a new second order transfer function to capture these features of the Bode plot.

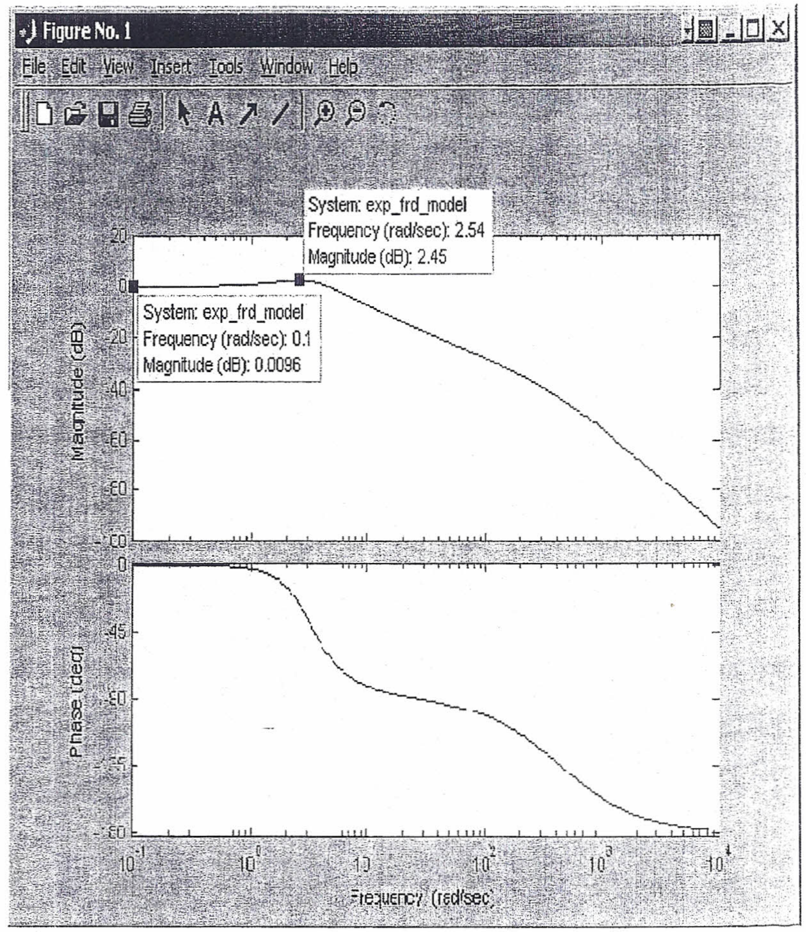


SYSTEM IDENTIFICATION of a high order system

To begin we will use the following second order transfer function model:

$$G(s) = k_{dc}\frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

Using the graph we can locate the frequency, w_p, and magnitude, Mp, of the peak and the DC gain, k_{dc}. By dragging the cursor along the plot of the graph, one can find the different values and frequencies. Find the magnitude at the lowest frequency of the graph which will be a good approximation of the DC gain. Also find the frequency and magnitude of the peak as shown.



SYSTEM IDENTIFICATION of a high order system

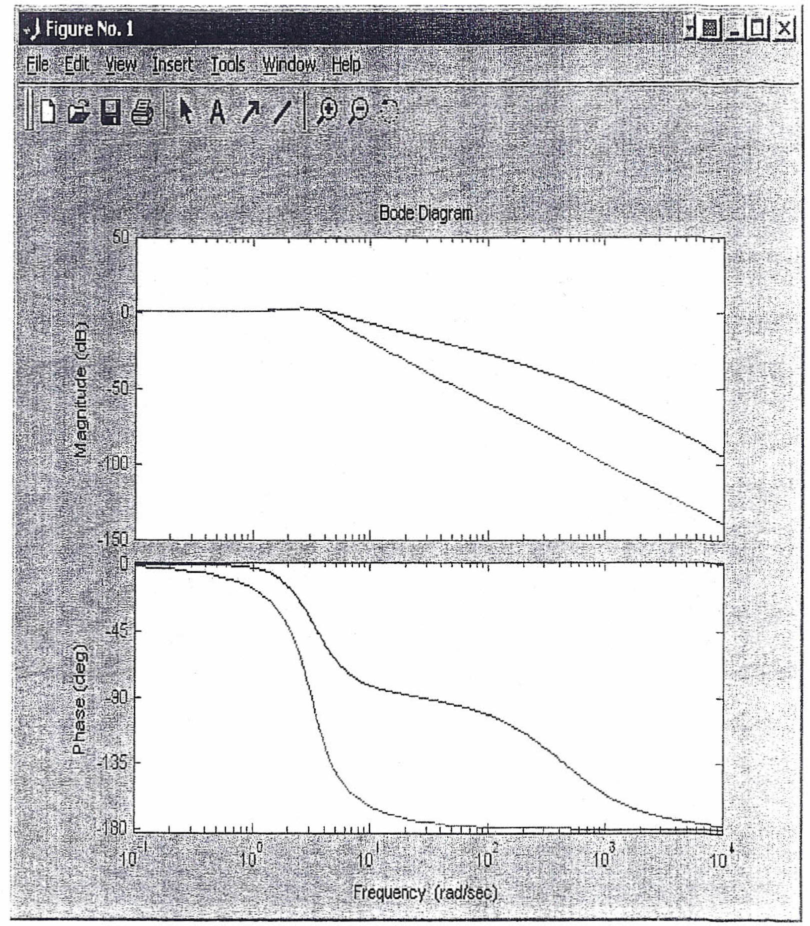
The relationship between features of the Bode plot and the parameters of the second order system are as follows:

$$k_{dc} = 10^{\left(zero\,frequency_{asymptote}\right)/20}$$

$$M_p = \frac{10^{(peak\_in\_db/20)}}{k_{dc}}$$

$$\zeta = \sqrt{\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{1}{M_p^2}}}$$

Where w_n is the natural frequency where the graph begins to turn and ζ is the damping ratio.
From the graph the k_{dc} = 1 (i.e. 0 dB), w_p = 2.54 rad/sec, and Mp = 1.33 (i.e. 2.45 dB).



SYSTEM IDENTIFICATION of a high order system

These differences suggest adding a so called lead compensator transfer function to the second order model. A lead compensator has the following form:

$$F(s) = \frac{\frac{s}{z}+1}{\frac{s}{p}+1}$$

The lead compensator has a zero at -z and a pole at -p, which are defined by:

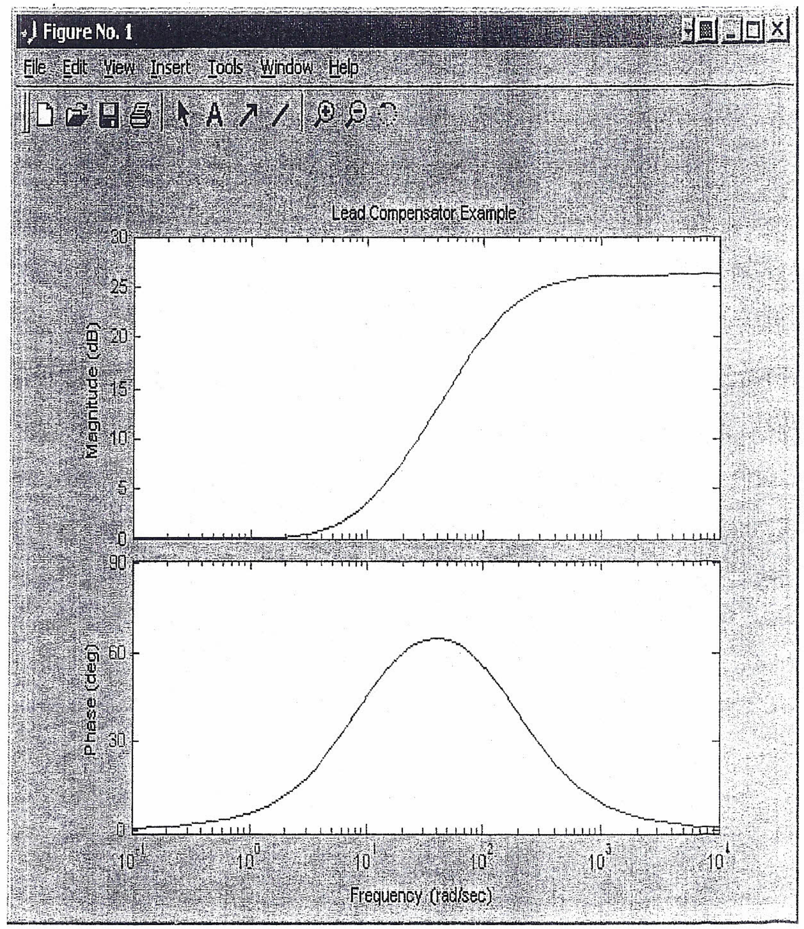
$$p = \frac{\omega_1}{\sqrt{a}} \qquad z = \omega_1 * \sqrt{a}$$

Here, is the amount of phase at the phase peak and is the frequency of the phase peak.

$$a = \frac{1 - sin\varphi}{1 + sin\varphi}$$

φ

ω_1



2.8

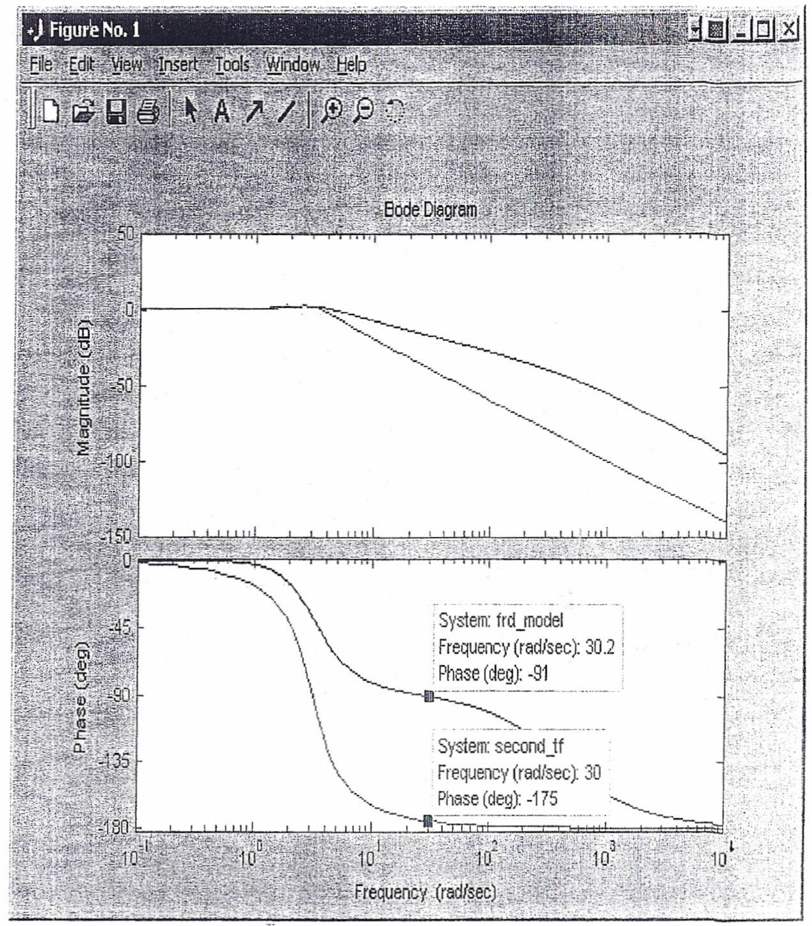
SYSTEM IDENTIFICATION of a high order system

The next step of the system identification process is to find the frequency at which the phase difference between the FRD model and the second order model is greatest. This frequency will be the of the lead compensator, and the ω difference in phase will be the of the lead compensator.

A good estimate for these values would be:

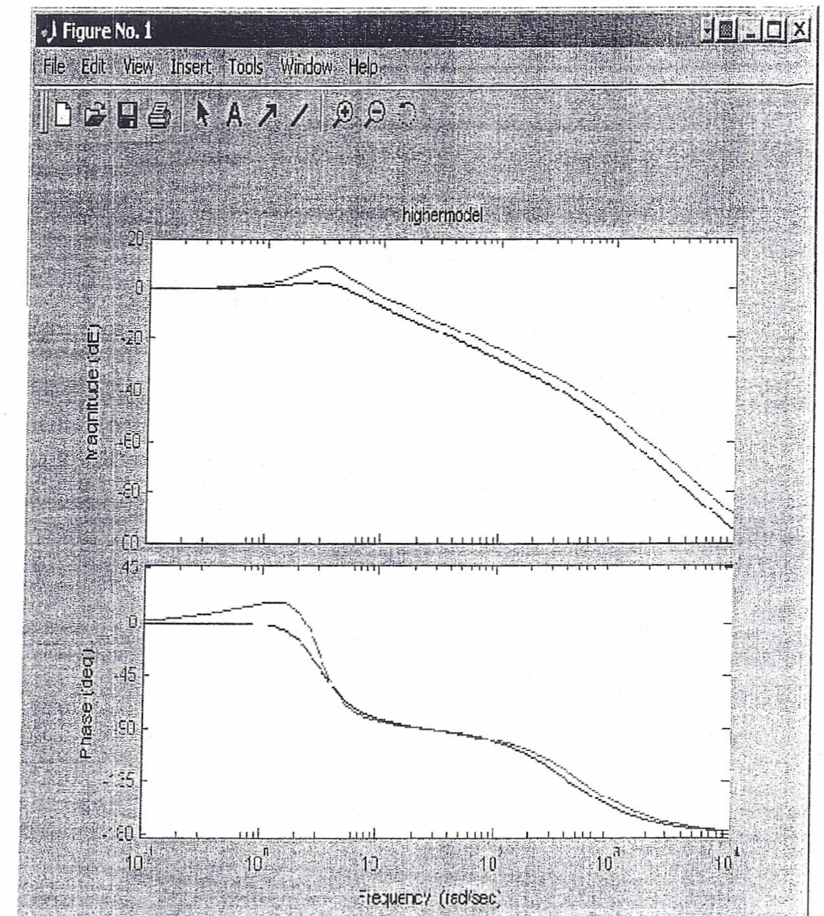
= 30 φ rad/sec, = (-91) - (-175) = 84 degrees.

ω_1 φ



SYSTEM IDENTIFICATION of a high order system

Then graph the Bode plot the of the second order transfer function multiplied with the lead compensator as shown. Notice that the peak is higher than expected. This is because of the influence of the lead compensator. Use just the resulting values of ζ =0.4127 and w_n=3.1282 without the equations as a inital values for further iterations.



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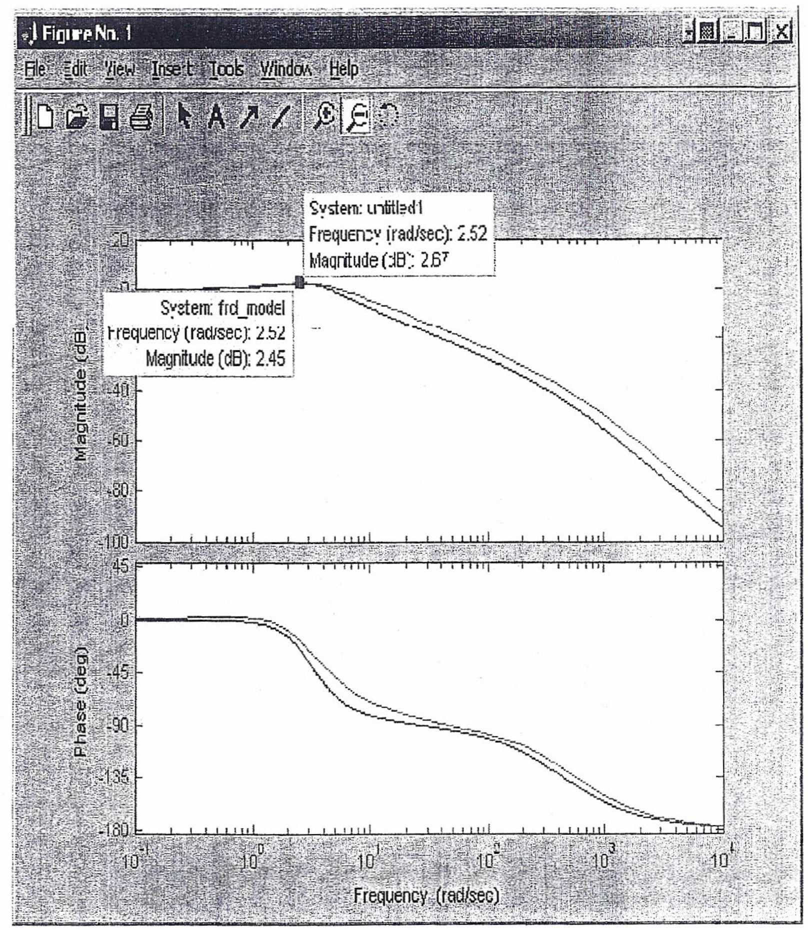
SYSTEM IDENTIFICATION of a high order system

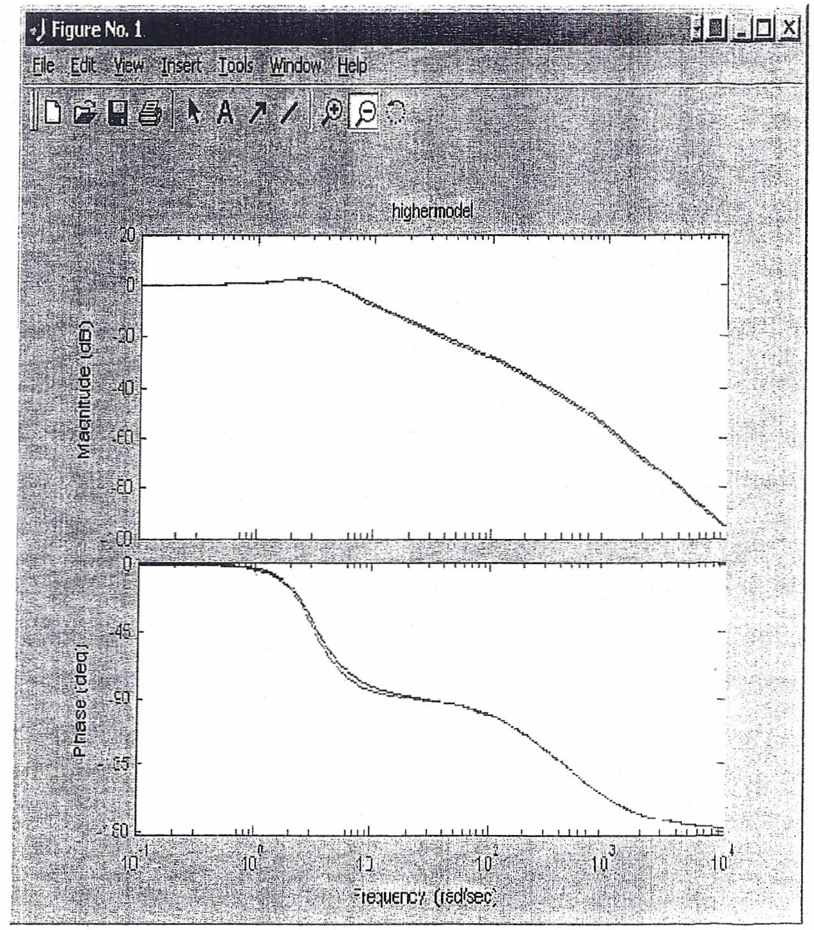
A good starting point for a iteration would be to find the difference between peaks of the FRD model and higher order model and multiply that value to ζ. Remember to convert from dB. Here the difference it found to be 6.11 dB (i.e. 2.02). Multiplying zeta by this value gives the following plots.

the peaks are nearly coincident and may be a sufficient model for many cases. Often times, matching the lower frequency values where the magnitude is highest is more important then matching the high ones where the magnitude drops off rapidly. Proceeding two iteration as follows results in an even better graph.



SYSTEM IDENTIFICATION of a high order system

Above, the peaks are nearly coincident and may be a sufficient model for many cases. Often times, matching the lower frequency values where the magnitude is highest is more important then matching the high ones where the magnitude drops off rapidly. Proceeding two iteration as follows results in an even better graph.



SYSTEM IDENTIFICATION of a high order system

```
% Iteration 1
zeta = zeta * 10^((8.56-2.45)/20)
second_tf = 1/((s/wn)^2 + 2*zeta*(s/wn) + 1);

% Iteration 2
wlead = 35;
philead = 81 * (pi / 180);
a = (1 - sin(philead))/(1+sin(philead));
zlead = wlead * sqrt(a);
plead = wlead / sqrt(a);
leadcomp = (s/zlead + 1)/(s/plead + 1);

% ...Iteration X
zeta = zeta / 10^((3.0)/20)
second_tf = 1/((s/wn)^2 + 2*zeta*(s/wn) + 1);
figure(1);
bode(frd_model, second_tf * leadcomp,{1, 10});
title('highermodel');
```

summary

- Nonparametric model estimation is simple and more efficient, but often less accurate, than parametric estimation.
- you can use a nonparametric model estimation method to obtain useful information about a system before applying parametric model estimation. For example, you can use nonparametric model estimation to determine whether the system requires preconditioning, what the time delay of the system is, what model order to select, and so on.
- You also can use nonparametric model estimation to verify parametric models. For example, you can compare the Bode plot of a parametric model with the frequency response of the nonparametric model